Topics in Category Theory:

Monads, Monoidal Categories and Actegories

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Abstract

We present a selection of results from Riehl's book [1], culminating in the exposition of monads, for which we present detailed proofs. Next, we introduce monoidal categories and actegories. We also use categorical machinery to give original proofs of some minor results.

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1 Introduction

In this report, we provide an exposition of central categorical concepts. In sections 2 and 3, we follow Riehl's presentation [1]. In particular, in section 2, we expose the definitions and results (without proof) of the categorical prerequisites for monads; with the exception of section 2.5, due to the present author, which applies Theorem 2.94 to give a characterization of the existence of left adjoints for set-valued functors, as well as a well-known recipe to compute coproducts in the category of groups. In section 3, we provide the main results and detailed proofs about monads; section 3.3, due to the present author, presents a uniqueness result about the Eilenberg-Moore category.

In section 4, we follow [2] and present the notion of a monoidal category, which is a categorification of the familiar notion of a monoid. A crucial construction from [3] was also exposed.

Finally, in section 5, we follow [4] and introduce the categorification of monoid actions: actegories - that is, actions of monoidal categories on categories.

2 Category Theory : Prerequisites for Monads

2.1 Categories, Functors and Natural Transformations

This subsection presents the fundamental mathematical objects of category theory.

Definition 2.1. A *category* $\mathcal{C}$ is a sextuple $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod}, \circ, \mathrm{id})$, where

- $\mathrm{Ob}\,\mathcal{C}$ is a collection, the elements of which are called *objects*;
- $\mathrm{Mor}\,\mathcal{C}$ is a collection, the elements of which are called *morphisms*;
- $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$ are assignments from the collection of morphisms to the collection of objects;
- if $\mathrm{Comp}\,\mathcal{C} := \{(g, f) \in \mathrm{Mor}\,\mathcal{C} \times \mathrm{Mor}\,\mathcal{C} \mid \mathrm{cod}(f) = \mathrm{dom}(g)\}$, there is an assignment $\circ : \mathrm{Comp}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ that to each composable pair of morphisms (g, f) assigns the composite morphism $g \circ f$; moreover, $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$;
- $\mathrm{id} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ is an injective assignment that to each object $x \in \mathrm{Ob}\,\mathcal{C}$ assigns the identity morphism id_x at x; it satisfies $\mathrm{dom}(\mathrm{id}_x) = \mathrm{cod}(\mathrm{id}_x) = x$;

satisfying two additional axioms:

- the *associativity axiom*, which requires that for any composable triple of morphisms f, g and h - that is, $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$ -

$$(h \circ g) \circ f = h \circ (g \circ f);$$

- the *identity axiom*, which states that for any morphism $f \in \mathrm{Mor}\,\mathcal{C}$

$$f \circ \mathrm{id}_{\mathrm{dom}(f)} = f = \mathrm{id}_{\mathrm{cod}(f)} \circ f.$$

Notation and Terminology 2.2. (i) For a morphism $f \in \mathrm{Mor}\,\mathcal{C}$, we picture f as an arrow $f : \mathrm{dom}(f) \to \mathrm{cod}(f)$ from its domain $\mathrm{dom}(f)$ to its codomain $\mathrm{cod}(f)$.

(ii) For two objects $c, d \in \mathrm{Ob}\,\mathcal{C}$, we denote by $\mathcal{C}(c, d)$ the collection of all morphisms $f : c \to d$ from c to d.

(iii) Regarding the data of a category, we will seldom be this verbose; often the assignments dom and cod are left implicit, for instance. Regarding the collection $\mathrm{Mor}\,\mathcal{C}$, one may rather define $\mathcal{C}(c, d)$ for any pair of objects $c, d \in \mathrm{Ob}\,\mathcal{C}$ as long as these collections are pairwise disjoint so that dom and cod are well-defined; if they aren't, they can always be made disjoint by tagging the necessary data to the morphisms in each collection. This, however, is rarely done in practice as it tends to hide the important data associated to a morphism.

(iv) Thanks to the associativity axiom, we may drop the parentheses and simply write $h \circ g \circ f$ for the common composite morphism.

(v) Finally, in order to improve readability, we will from now on adopt the uniform notation $c \in \mathcal{C}$ and $f \in \mathcal{C}$ for both objects and morphisms, whenever the context enables us to distinguish them.

Remark 2.3. Note that we intentionally employ the vague terms "collection" and "assignment" instead of "set" and "function" as we allow categories to have more than a set's worth of objects or morphisms. Specific terminology is used when the collection of objects and/or morphisms do form a set.

Definition 2.4. A category $\mathcal{C}$ is *small* if its collection of morphisms $\operatorname{Mor}\mathcal{C}$ - and hence also its collection of objects $\operatorname{Ob}\mathcal{C}$ by injectivity of id - is a set.

A category $\mathcal{C}$ is *locally small* if for any two objects $c, d \in \operatorname{Ob}\mathcal{C}$, the collection $\mathcal{C}(c,d)$ is a set.

Remark 2.5. For any locally small category $\mathcal{C}$ and object $c \in \mathcal{C}$, $\operatorname{End}_{\mathcal{C}}(c) := \mathcal{C}(c,c)$ admits a monoid structure, where the monoid law is composition of endomorphisms and the identity is id_c

Example 2.6. Here are examples of categories:

(i) the category **Set** of sets and functions between them, with the usual composition of functions and the identity function at a set;

(ii) the category $\mathbf{Set}_*$ of pointed sets and pointed maps;

(iii) the category **Top** of topological spaces and continuous maps;

(iv) the category $\mathbf{Top}_*$ of pointed spaces and pointed continuous maps;

(v) the category $\mathbf{Htpy}_*$ whose objects are pointed topological spaces and whose morphisms are homotopy classes of pointed continuous maps.

Among algebraic categories, the categories

(vi) **Monoid** of monoids and monoid homomorphisms;

(vii) **Group** of groups and group homomorphisms;

(viii) **Ring** of rings and ring homomorphisms;

(ix) for rings R and S, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_S$ and ${}_R\mathbf{Mod}_S$ of left R-modules, right S-modules and (R,S)-bimodules respectively and module homomorphisms;

(x) taking $R = k$ for a field k, there is the category $\mathbf{Vect}_k$ of vector spaces and linear maps.

There are also constructions such as

(xi) the category $\mathbf{B}M$ for a monoid $(M, *, e)$, with a single object $\bullet$, whose morphisms $\bullet \to \bullet$ are elements of M, composition given by multiplication in M and the identity on $\bullet$ is e;

(xii) the category $\mathbf{Disc}(X)$ for a set X, whose objects are elements of X and the only morphisms are the required identity morphisms on each object;

(xiii) any preorder $(P, \leq)$ defines a category whose objects are the elements of P, and there is a (single) morphism $x \to y$ if and only if $x \leq y$;

(xiv) for a ring R, there is a category $\mathbf{Mat}_R$ whose objects are positive integers 1, 2, 3, $\cdots$ and a morphism $n \to m$ is an $m \times n$ matrix $A \in \mathcal{M}_{m\times n}(R)$; composition is given my matrix multiplication, and the identity at n is the identity matrix Id_n.

Definition 2.7. A morphism $f : c \to d \in \mathcal{C}$ is an *isomorphism* if there exists a morphism $g : d \to c$ pointing in the other direction so that $g \circ f = \mathrm{id}_c$ and $f \circ g = \mathrm{id}_d$; in this case, we write $f : c \xrightarrow{\cong} d$. When such a morphism g exists, it is unique and is called the *inverse morphism* of f, written f^{-1}.

Two objects $c, d \in \mathcal{C}$ are *isomorphic* if there exists an isomorphism $f : c \xrightarrow{\cong} d$.

Example 2.8. In the category **Set**, isomorphisms are bijections between sets.

Definition 2.9. A *groupoid* is a category in which every morphism is an isomorphism.

Example 2.10. For a group G, the category $\mathbf{B}G$ is a groupoid because each morphism $g : \bullet \to \bullet$ has inverse $g^{-1} : \bullet \to \bullet$.

We may be interested in inducing the structure of a category to a smaller collection, analogous to the notion of a subgroup of a group in group theory, or the notion of a subspace of a topological space in topology.

Definition 2.11. Let $\mathcal{C}$ be a category. Suppose we have a collection of objects $\mathrm{Ob}\,\mathcal{D} \subset \mathrm{Ob}\,\mathcal{C}$ and a collection of morphisms $\mathrm{Mor}\,\mathcal{D} \subset \mathrm{Mor}\,\mathcal{C}$ so that

- $\forall d \in \mathrm{Ob}\,\mathcal{D}$, the identity at d remains in the collection of morphisms: $\mathrm{id}_d \in \mathrm{Mor}\,\mathcal{D}$;
- $\forall f, g \in \mathrm{Ob}\,\mathcal{D}$ that are composable, the composite remains in the collection of morphisms : $g \circ f \in \mathrm{Mor}\,\mathcal{D}$.

Then by restricting the data of the category $\mathcal{C}$, we obtain a new category $\mathcal{D}$, called a *subcategory* of $\mathcal{C}$.

A subcategory $\mathcal{D}$ is called *full* if

$$\forall c, d \in \mathrm{Mor}\,\mathcal{D},\ f \in \mathcal{C}(c, d) \implies f \in \mathrm{Mor}\,\mathcal{D}$$

In other words, a full subcategory is constructed by choosing a collection of objects from the original category $\mathcal{C}$ and keeping all existing morphisms between them.

There are many examples of subcategories:

Example 2.12.
(i) there is a full subcategory **Fin** of **Set**, which consists of all finite sets and functions between them. There is also a subcategory (which is not full) $\mathbf{Fin}_{\mathrm{iso}}$ of finite sets and bijections between them;

(ii) the category **Ab** of abelian groups and abelian group homomorphisms is a full subcategory of **Group**;

(iii) the category **Field** of fields and field homomorphisms is a full subcategory of **Ring**;

(iv) the category **cHaus** of compact Hausdorff spaces and continuous maps between them is a full subcategory of **Top**;

(v) the category $\mathbf{Vect}_k^{\mathrm{fd}}$ of finite-dimensional vector spaces over a field k and linear maps between them is a full subcategory of $\mathbf{Vect}_k$;

(vi) for any category $\mathcal{C}$, there is a maximal subgroupoid $\mathbf{Iso}_{\mathcal{C}}$, whose objects are all the objects of $\mathcal{C}$, and whose morphisms are only the isomorphisms in $\mathcal{C}$.

We now introduce the concept of duality, which enables us to obtain two (dual) notions/statements from one notion/statement.

Definition 2.13. For any category $\mathcal{C}$, there is an *opposite category* $\mathcal{C}^{op}$ defined by

- $\operatorname{Ob}\mathcal{C}^{op} = \operatorname{Ob}\mathcal{C}$;
- $\operatorname{Mor}\mathcal{C}^{op} = \operatorname{Mor}\mathcal{C}$;
- $\operatorname{dom}^{op} = \operatorname{cod}$ and $\operatorname{cod}^{op} = \operatorname{dom}$;
- note that in particular $(f, g) \in \operatorname{Comp}\mathcal{C}^{op} \iff (g, f) \in \operatorname{Comp}\mathcal{C}$; we define $\circ^{op}(f, g) = \circ(g, f)$;
- $\mathrm{id}^{op} = \mathrm{id}$.

Thus the opposite category is the category with the same objects, in which each arrow gets reversed: the new domain is the old codomain, and the new codomain is the old domain. The morphism in $\mathcal{C}^{op}$ corresponding to $f : c \to d \in \mathcal{C}$ is denoted by $f^{op} : d \to c \in \mathcal{C}^{op}$.

It is possible to generalize the notion of "injective map" and "surjective map" in **Set**.

Definition 2.14. A morphism in a category $f : x \to y$ is

(i) a *monomorphism*, depicted by $f : x \rightarrowtail y$, if for any parallel pair of morphisms $h, k : w \rightrightarrows x$, $f \circ h = f \circ k$ implies $h = k$;

(ii) an *epimorphism*, depicted by $f : x \twoheadrightarrow y$, if for any parallel pair of morphisms $h, k : y \rightrightarrows z$, $h \circ f = k \circ f$ implies $h = k$.

Example 2.15. In **Set**, monomorphisms are injective maps and epimorphisms are surjective maps. In this special case, isomorphisms are exactly those morphisms that are both monomorphisms and epimorphisms.

Manipulating the definitions, we see that these notions are dual: a morphism $f : x \to y$ is a monomorphism in $\mathcal{C}$ if and only if $f^{op} : y \to x$ is an epimorphism in $\mathcal{C}^{op}$.

Definition 2.16. A morphism in a category $f : x \to y$ is

(i) a *split monomorphism* if there exists a morphism $g : y \to x$ so that $g \circ f = \mathrm{id}_x$

(ii) a *split epimorphism* if there exists a morphism $h : y \to x$ so that $f \circ h = \mathrm{id}_y$

Again one can see that these notions are dual. One can prove that in any category $\mathcal{C}$, any split monomorphism is a monomorphism. Applying this result to $\mathcal{C}^{op}$, we obtain a "proof by duality" that also any split epimorphism in an arbitrary category is an epimorphism. The power of duality thus resides in its capacity of generating dual definitions, and with it, dual results.

We can compare categories using "maps" them.

Definition 2.17. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a pair $(F_{\mathrm{Ob}}, F_{\mathrm{Mor}})$, where $F_{\mathrm{Ob}} : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ and $F_{\mathrm{Mor}} : \operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}$ are assignments between objects and morphisms respectively, such that

- for any morphism $f : c \to c' \in \mathcal{C}$, we have $F_{\mathrm{Mor}}(f) : F_{\mathrm{Ob}}(c) \to F_{\mathrm{Ob}}(c')$; or in other words, the following diagrams commute:

$$\begin{array}{ccc} \operatorname{Mor}\mathcal{C} & \xrightarrow{F_{\mathrm{Mor}}} & \operatorname{Mor}\mathcal{D} \\ {\scriptstyle \mathrm{dom}_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{dom}_{\mathcal{D}}} \\ \operatorname{Ob}\mathcal{C} & \xrightarrow[F_{\mathrm{Ob}}]{} & \operatorname{Ob}\mathcal{D} \end{array} \qquad \begin{array}{ccc} \operatorname{Mor}\mathcal{C} & \xrightarrow{F_{\mathrm{Mor}}} & \operatorname{Mor}\mathcal{D} \\ {\scriptstyle \mathrm{cod}_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{cod}_{\mathcal{D}}} \\ \operatorname{Ob}\mathcal{C} & \xrightarrow[F_{\mathrm{Ob}}]{} & \operatorname{Ob}\mathcal{D} \end{array} ;$$

- for any $c \in \operatorname{Ob}\mathcal{C}$, we have $F_{\mathrm{Mor}}(\mathrm{id}_c) = \mathrm{id}_{F_{\mathrm{Ob}}(c)}$; visually:

$$\begin{array}{ccc} \operatorname{Ob}\mathcal{C} & \xrightarrow{F_{\mathrm{Ob}}} & \operatorname{Ob}\mathcal{D} \\ {\scriptstyle \mathrm{id}_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{D}}} \\ \operatorname{Mor}\mathcal{C} & \xrightarrow[F_{\mathrm{Mor}}]{} & \operatorname{Mor}\mathcal{D} \end{array} \quad ;$$

- for any composable pair of morphisms $c \xrightarrow{f} c' \xrightarrow{g} c''$, we have $F_{\mathrm{Mor}}(g \circ f) = F_{\mathrm{Mor}}(g) \circ F_{\mathrm{Mor}}(f)$; equivalently, the following square commutes:

$$\begin{array}{ccc} \operatorname{Comp}\mathcal{C} & \xrightarrow{F_{\mathrm{Mor}} \times F_{\mathrm{Mor}}} & \operatorname{Comp}\mathcal{D} \\ {\scriptstyle \circ_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \circ_{\mathcal{D}}} \\ \operatorname{Ob}\mathcal{C} & \xrightarrow[F_{\mathrm{Mor}}]{} & \operatorname{Ob}\mathcal{D} \end{array} \quad .$$

Definition 2.18. Given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, there is a *composite functor* $G \circ F : \mathcal{C} \to \mathcal{E}$ defined by $((G \circ F)_{\mathrm{Ob}}, (G \circ F)_{\mathrm{Mor}}) = (G_{\mathrm{Ob}} \circ F_{\mathrm{Ob}}, G_{\mathrm{Mor}} \circ F_{\mathrm{Mor}})$. There is also an *identity functor* on $\mathcal{C}$, $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, defined by $(\mathrm{id}_{\operatorname{Ob}\mathcal{C}}, \mathrm{id}_{\operatorname{Mor}\mathcal{C}})$.

Note that composition is associative, and the identity is unital. In particular, there is a category **Cat** whose objects are small categories and morphisms are functors. If we want to include more categories, there is also a category **CAT** whose objects are locally small categories.

The notion of a functor is ubiquitous:

Example 2.19. (i) there is an endofunctor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ that carries a set A to its power set $\mathcal{P}A$ (the set of all subsets of A), and a function $f : A \to B$ to the direct image function $f_* : \mathcal{P}A \to \mathcal{P}B$ defined by $f_*(A') := \{f(x) \mid x \in A'\} \subset B$ for $A' \subset A$;

(ii) there is a functor $\mathrm{Path} : \mathbf{Top} \to \mathbf{Set}$ that carries a space X to the set of all paths $\mathbf{Top}(I, X)$ in X ($I = [0, 1]$ is the unit interval), and a continuous map $f : X \to Y$ to the assignment $\gamma : I \to X \in \mathrm{Path}(X) \mapsto f \circ \gamma : I \to Y \in \mathrm{Path}(Y)$;

(iii) the fundamental group functor $\pi_1 : \mathbf{Htpy}_* \to \mathbf{Group}$ carries a pointed space (X, x) to its fundamental group $\pi_1(X, x)$, and the homotopy class of a pointed map $[f : (X, x) \to (Y, y)]$ to the group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ which sends the homotopy class of a loop $[\gamma]$ to $[f \circ \gamma]$. This functor provides a connection between topology and group theory and can be used to prove many results about the fundamental group;

(iv) for a subcategory $\mathcal{D}$ of a category $\mathcal{C}$, there is an inclusion functor $I : \mathcal{D} \hookrightarrow \mathcal{C}$. For instance, the inclusion functors $\mathbf{Ab} \hookrightarrow \mathbf{Group}$, $\mathbf{Field} \hookrightarrow \mathbf{Ring}$ and $\mathbf{cHaus} \hookrightarrow \mathbf{Top}$;

(v) for a group G, a left G-set $(X, G \curvearrowright X)$ can be viewed as a functor $X : \mathbf{B}G \to \mathbf{Set}$: it is a functorial assignment that carries each $g \in G$ to a bijection (by Lemma 2.24) $g* : X \to X$;

(vi) there are functors $\mathrm{Ob}, \mathrm{Mor} : \mathbf{Cat} \to \mathbf{Set}$ that carry a category $\mathcal{C}$ to its set of objects $\operatorname{Ob}\mathcal{C}$ and set of morphisms $\operatorname{Mor}\mathcal{C}$ resp., and a functor $F : \mathcal{C} \to \mathcal{D}$ between categories to the components F_{Ob} and F_{Mor} resp.

There is a class of so-called "forgetful" functors, often denoted by U, which forget some structure or properties. Examples include:

(vii) $U : \mathbf{Set}_* \to \mathbf{Set}$ that sends a pointed set (X, x) to X and a pointed map $f : (X, \mathrm{x}) \to (Y, \mathrm{y})$ to the underlying map $f : X \to Y$ between the sets;

(viii) $U : \mathbf{Group} \to \mathbf{Set}$ that carries a group to its underlying set, and a group homomorphism to the corresponding map between underlying sets;

(ix) : $U : \mathbf{Ring} \to \mathbf{Ab}$ that forgets the monoid structure of a ring and only keeps the abelian group structure;

(x) we may also view the inclusion $I : \mathbf{Ab} \hookrightarrow \mathbf{Group}$ as a forgetful functor, forgetting the commutativity property of an abelian group;

(xi) a functor between preorders is an order-preserving map $F : (P, \leq) \to (Q, \leq)$; that is,

$$p \leq p' \implies Fp \leq Fp'.$$

Notation and Terminology 2.20. In practice, to avoid heavy notation, we will denote both the map between objects and the map between morphisms simply by F. Moreover, we will drop the parentheses and write Fc or Ff for $c \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathrm{Mor}\,\mathcal{C}$.

Definition 2.21. A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$. Explicitly, it consists of

- an object Fc for each $c \in \mathcal{C}$;
- a morphism $Ff : Fc' \to Fc$ for each morphism $f : c \to c'$;

satisfying two functoriality axioms:

- for any composable pair f, g in $\mathcal{C}$, $F(g \circ f) = Ff \circ Fg$;
- for all $c \in \mathcal{C}$, $F(\mathrm{id}_c) = \mathrm{id}_{Fc}$.

Remark 2.22. A functor $F : \mathcal{C} \to \mathcal{D}$ induces in an evident way a functor from $\mathcal{C}^{op}$ to $\mathcal{D}^{op}$, that we denote by $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$.

Contravariant functors naturally arise in various contexts:

Example 2.23. (i) there is a contravariant power set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Set}$ that carries a set A to its power set $\mathcal{P}A$, and a map $f : A \to B$ to the preimage map $f^{-1} : \mathcal{P}B \to \mathcal{P}A$ defined by $f^{-1}(B') = \{x \in A \mid f(x) \in B'\} \subset A$ for $B' \subset B$;

(ii) the topology functor $\mathcal{T} : \mathbf{Top} \to \mathbf{Set}$ that to each topological space $(X, \mathcal{T}_X)$ associates the topology $\mathcal{T}_X$ on X, and to a continuous function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ associates the preimage map $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$, which is well-defined by the very definition of a continuous map;

(iii) the dual functor $(-)^* : \mathbf{Vect}_k \to \mathbf{Vect}_k$ carries a vector space V to its dual $V^* := \mathrm{Hom}_k(V, k)$, and a linear map $\varphi : V \to W$ to the dual map $\varphi^* : W^* \to V^*$, which sends $l \in W^*$ to $l \circ \varphi \in V^*$.

Using all of the axioms of a functor, one can prove the following lemma.

Lemma 2.24. *Functors preserve isomorphisms. That is, for a functor $F : \mathcal{C} \to \mathcal{D}$, if $f \in \mathcal{C}$ is an isomorphism, then $Ff \in \mathcal{D}$ is an isomorphism.*

Example 2.25. For example, by functoriality of $\pi_1 : \mathbf{Htpy}_* \to \mathbf{Group}$, since isomorphisms in $\mathbf{Htpy}_*$ are precisely homotopy classes of pointed homotopy equivalences, it follows that a pointed homotopy equivalence $f : (X, x) \xrightarrow{\sim} (Y, y)$ induces an isomorphism $f_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, y)$ of fundamental groups

Definition 2.26. For any categories $\mathcal{C}$, $\mathcal{D}$ there is a *product category* $\mathcal{C} \times \mathcal{D}$ defined as follows:

- objects are pairs (c, d), where $c \in \mathcal{C}$ and $d \in \mathcal{D}$;
- morphisms $(f, g) : (c, d) \to (c', d')$ are pairs of morphisms, where $f : c \to c' \in \mathcal{C}$ and $g : d \to d' \in \mathcal{D}$;
- composition is computed in each coordinate: $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$;
- the identity at (c, d) is $(\mathrm{id}_c, \mathrm{id}_d)$.

Remark 2.27. For a bifunctor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ - that is, an ordinary functor whose domain category is a product of categories - it is possible to fix one variable to obtain a new functor.

For $a \in \mathcal{A}$, there is a functor

$$F(a, -) : \mathcal{B} \xrightarrow{\{a\}\times -} \mathcal{A} \times \mathcal{B} \xrightarrow{F} \mathcal{C},$$

where the inclusion

$$\{a\} \times - : \mathcal{B} \to \mathcal{A} \times \mathcal{B}$$

sends $b \in B$ to $(a, b) \in \mathcal{A} \times \mathcal{B}$ and $f : b \to b' \in \mathcal{B}$ to $(\mathrm{id}_a, f) \in \mathcal{A} \times \mathcal{B}$.

Similarly, for each $b \in B$, there is a functor $F(-, b) : \mathcal{A} \to \mathcal{C}$.

Definition 2.28. If $\mathcal{C}$ is a locally small category, there is a *two-sided represented functor* $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ defined by

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathcal{C}(-,-)} & \mathbf{Set} \\ (x, y) & \longmapsto & \mathcal{C}(x, y) \\ {\scriptstyle f}\uparrow\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_* \circ - \circ f^*} \\ (w, z) & \longmapsto & \mathcal{C}(w, z) \end{array} .$$

By Remark 2.27, for any $x \in \mathcal{C}$, there are functors obtained by fixing one variable:

$$\mathcal{C}(x, -) : \mathcal{C} \to \mathbf{Set}$$

and

$$\mathcal{C}(-, x) : \mathcal{C}^{op} \to \mathbf{Set}.$$

It is also possible to compare functors between a fixed pair of categories via "maps" between them.

Definition 2.29. Given categories $\mathcal{C}$, $\mathcal{D}$ and a parallel pair of functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ is an assignment $\alpha : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ so that for each $c \in \mathrm{Ob}\,\mathcal{C}$, the component $\alpha_c : Fc \to Gc$. Moreover, for all $f : c \to c' \in \mathrm{Mor}\,\mathcal{C}$, the following naturality square commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & Gc' \end{array} .$$

A natural isomorphism is a natural transformation such that each component α_c is an isomorphism.

Notation and Terminology 2.30. A natural transformation is often depicted as

$$\mathcal{C} \;\overset{F}{\underset{G}{\Downarrow \alpha}}\; \mathcal{D} \quad .$$

Here are examples of natural transformations between functors:

Example 2.31. (i) there is a natural isomorphism $\mathrm{ev} : \mathrm{id}_{\mathbf{Vect}_k} \Rightarrow (-)^{**}$, whose component at $V \in \mathbf{Vect}_k$ is the linear map $\mathrm{ev}_V : V \xrightarrow{\cong} V^{**}$ that carries $v \in V$ to

$$\mathrm{ev}_{V,v} : l \in V^* \mapsto l(v) \in k;$$

(ii) there is a natural transformation $\eta : \mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$, whose component at $A \in \mathbf{Set}$ is

$$\eta_A : a \in A \mapsto \{a\} \in \mathcal{P}A;$$

(iii) given two left G-sets $X, Y : \mathbf{B}G \to \mathbf{Set}$, a natural transformation $f : X \Rightarrow Y$ is exactly a G-equivariant map between underlying sets $f : X \to Y$: for every $g \in G$, the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g*}\downarrow & & \downarrow{\scriptstyle g*} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes. Evaluating both composites at an element $x \in X$, commutativity of the square reads

$$f(g \cdot x) = g \cdot f(x);$$

(iv) for a topological example, there are natural transformations $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Path} \to U$, whose components at a space X

$$\mathrm{ev}_{0,X}, \mathrm{ev}_{1,X} : \mathrm{Path}(X) \to X$$

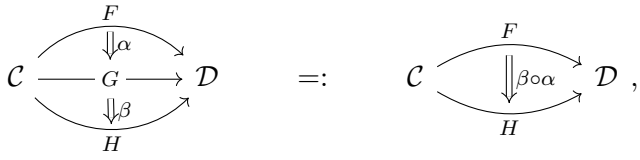
evaluate a path $\gamma : I \to X$ at 0 and 1 respectively;

(v) there are suggestively-named natural transformations $\mathrm{id} : \mathrm{Ob} \Rightarrow \mathrm{Mor}$ and $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor} \Rightarrow \mathrm{Ob}$;

(vi) given functors $F, G : (P, \leq) \to (Q, \leq)$, there is a natural transformation $F \Rightarrow G$ if and only if

$$Fp \leq Gp, \forall p \in P.$$

Definition 2.32. (vertical composition) Given categories, functors and natural transformations as depicted on the left of the diagram below



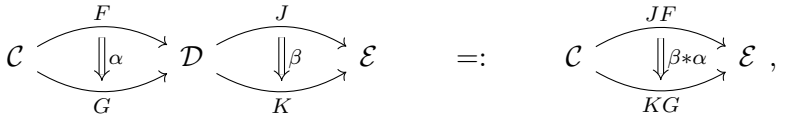
we can form the *composite natural transformation* $\beta \circ \alpha : F \Rightarrow H$, whose component at $c \in \mathcal{C}$ is defined by $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$.

There is an identity natural transformation on F, $\mathrm{id}_F : F \Rightarrow F$, whose component at $c \in \mathcal{C}$ is $\mathrm{id}_{Fc} : Fc \to Fc$.

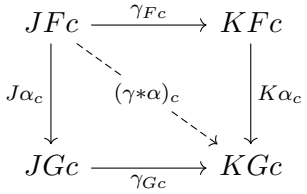
Again, composition is associative, and the identity is unital. In particular, there is a *functor category* $\mathcal{D}^{\mathcal{C}}$ whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms between them are natural transformations.

Remark 2.33. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $\mathcal{D}^{\mathcal{C}}$ is again locally small.

Definition 2.34. (horizontal composition) Given categories, functors and natural transformations as depicted below on the left

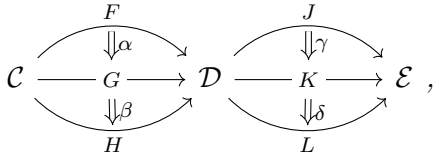


there is a natural transformation $\gamma * \alpha : JF \to KG$ whose component at $c \in \mathcal{C}$ is defined by either composite

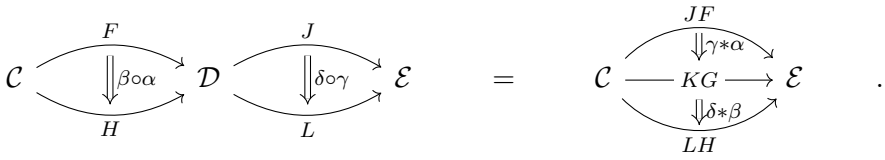


Remark 2.35. When $F = G$ and $\alpha = \mathrm{id}_F$, the horizontal composite gives us a natrual transformation $\gamma F : JF \Rightarrow KF$. Similarly, choosing $J = K$ and $\gamma = \mathrm{id}_J$, we obtain $J\alpha : JF \Rightarrow JG$. These natural transformations are obtained via a process called *whiskering* of a natural transformation.

Lemma 2.36 (exchange law for natural transformations)**.** *Given functors and natural transformations*



the natural transformation $JF \Rightarrow LH$ defined by first composing vertically and then composing horizontally equals the natural transformation defined by first composing horizontally and then composing vertically:



Definition 2.37. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. It is denoted by $F : \mathcal{C} \xrightarrow{\cong} \mathcal{D}$

Example 2.38. A very important example is the following Fubini-type relation:

$$\mathcal{C}^{\mathcal{A}\times\mathcal{B}} \cong (\mathcal{C}^{\mathcal{B}})^{\mathcal{A}} \tag{2.1}$$

for any categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$. We describe its action on objects explicitely. A functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is mapped to the functor $\tilde{F} : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$, where $\tilde{F}(a) = F(a, -)$, $\tilde{F}(f : a \to a')_b = F(f, \mathrm{id}_b) : F(a, b) \to F(a', b)$. This process is also known as currying.

There is a weaker notion which defines when two categories are "essentially the same".

Definition 2.39. An *equivalence of categories* consists of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ so that $\mathrm{id}_{\mathcal{C}} \cong GF$ and $FG \cong \mathrm{id}_{\mathcal{D}}$. A functor $F : \mathcal{C} \to \mathcal{D}$ defines an *equivalence of categories* if it extends to an equivalence of categories, in which case it is dentoed by $F : \mathcal{C} \xrightarrow{\simeq} \mathcal{D}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, written $\mathcal{C} \simeq \mathcal{D}$, if there is an equivalence between them.

Definition 2.40. A functor $F : \mathcal{C} \to \mathcal{D}$ is

- *full* if for any $x, y \in \mathcal{C}$, the map $f \in \mathcal{C}(x,y) \mapsto Ff \in \mathcal{D}(Fx, Fy)$ is surjective;
- *faithful* if for any $x, y \in \mathcal{C}$, the map $f \in \mathcal{C}(x,y) \mapsto Ff \in \mathcal{D}(Fx, Fy)$ is injective;
- *fully faithful* if it is both full and faithful; in other words, for any $x, y \in \mathcal{C}$, the map $f \in \mathcal{C}(x,y) \mapsto Ff \in \mathcal{D}(Fx, Fy)$ is bijective;
- *essentially surjective on objects* if for each $d \in \mathcal{D}$, $\exists c \in \mathcal{C}$ so that $Fc \cong d$.

Definition 2.41. A *concrete* category is a category $\mathcal{C}$ equipped with a faithful functor $U : \mathcal{C} \to \mathbf{Set}$, called the *forgetful functor* or *underlying-set functor*.

Definition 2.42. A functor $F : \mathcal{C} \to \mathcal{D}$ defines an *embedding* if its is fully faithful and injective on objects.

Theorem 2.43. *A functor $F : \mathcal{C} \to \mathcal{D}$ defines an equivalence if and only if it is fully faithful and essentially surjective on objects.*

Remark 2.44. While the notion of isomorphism of categories is very strong, requiring the two categories to be the "same" up to a relabelling of the objects and morphisms (like the notion of a bijection of sets), the notion of equivalence of categories only requires the categories to be "essentially the same". To illustrate this, me may chose our favourite category $\mathcal{C}$; if we now augment it to a category $\mathcal{C}'$ by adding objects in such a way that they are isomorphic to the original objects of $\mathcal{C}$ (in particular, we need to add non-identity morphisms), we would obtain an equivalent category (via the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}'$, by Theorem 2.43), but not necessarily an isomorphic one (certainly not if $\mathcal{C}$ has finitely many objects and morphisms).

Example 2.45. There is an equivalence $\mathbf{Mat}_k \simeq \mathbf{Vect}_k^{\mathrm{fd}}$, exhibited by the functor $k^- : \mathbf{Mat}_k \to \mathbf{Vect}_k^{\mathrm{fd}}$ that sends at integer n to the n-dimenstional k-vector space k^n, and a matrix an $m \times n$ matrix $A : n \to m$ to the corresponding linear map $A : k^n \to k^m$. From linear algebra, we know that k^- is fully faithful and essentially surjective on objects (each n-dimensional vector space V is isomorphic to k^n). Thus by Theorem 2.43, it defines an equivalence of categories. This provides a connection between "concrete" and "abstract" linear algebra.

The following proposition asserts that the notion of equivalence behaves as expected with natural isomorphisms between functors.

Proposition 2.46. *Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be a parallel pair of naturally isomorphic functors $F \cong G$. If F is an equivalence, then G is also an equivalence.*

2.2 Universal properties, representability and the Yoneda lemma

The aim of this subsection is to give a precise meaning to the notion of a "universal property", along with the categorical machinery surrounding it.

Definition 2.47. Let $\mathcal{C}$ be a locally small category. A *representation* for a covariant functor $F : \mathcal{C} \to \mathbf{Set}$ is an object $c \in \mathcal{C}$ togther with a natural isomorphism $\alpha : \mathcal{C}(c, -) \overset{\cong}{\Longrightarrow} F$.

Dually, if $F : \mathcal{C}^{op} \to \mathbf{Set}$ is a contravariant, a *representation* for F is given by an object $c \in \mathcal{C}$ togther with a natural isomorphism $\alpha : \mathcal{C}(-, c) \overset{\cong}{\Longrightarrow} F$.

In this case, we say that F is *representable*, and that c *represents* F.

Here are some examples of representations for covariant functors:

Example 2.48. (i) for a category $\mathcal{C}$, any initial object $c \in \mathcal{C}$ represents the constant functor at the singleton $\Delta * : \mathcal{C} \to \mathbf{Set}$ as each hom-set $\mathcal{C}(c, x)$ is a singleton; dually, a terminal object represents $\Delta * : \mathcal{C}^{op} \to \mathbf{Set}$;

(ii) the singleton set $*$ represents $\mathrm{id}_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ because, a function $* \xrightarrow{x} X$ corresponds to an element of the set $x \in X$;

(iii) the singleton space $*$ represents the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ since a continuous map $* \xrightarrow{x} (X, \mathcal{T})$ is uniquely determined by an element of the underlying set of the space $x \in X$;

(iv) the group $\mathbb{Z}$ (the free group on one generator, see Proposition 2.96) represents the forgetful functor $U : \mathbf{Group} \to \mathbf{Set}$. Indeed, for any group G, there is a bijection $\mathbf{Group}(\mathbb{Z}, G) \cong UG$; a homomorphism $\mathbb{Z} \to G$ can be evaluated at 1 to produce an element $g \in G$, which in turn defines the exponential homomorphism $g^- : \mathbb{Z} \to G$;

(v) in a similar way, the ring $\mathbb{Z}[t]$ represents $U : \mathbf{Ring} \to \mathbf{Set}$;

(vi) the terminal category $\mathbf{1}$ - that is, the discrete category on the singleton, with one single object and identity morphism - represents $\mathrm{Ob} : \mathbf{Cat} \to \mathbf{Set}$: to specify an object $c \in \mathcal{C}$ in a category is the same as giving a functor $c : \mathbf{1} \to \mathcal{C}$;

(vii) the poset category on $\{\{0,1\}, 0 \leq 1\}$ with 2 distinct objects and a single non-identity morphism $\bullet \to \bullet$, denoted by 2, represents $\mathrm{Mor} : \mathbf{Cat} \to \mathbf{Set}$. Indeed, a morphism $f : c \to c' \in \mathcal{C}$ corresponds to a functor $f : 2 \to \mathbf{Cat}$.

There are also examples in the contravariant case:

Example 2.49. (i) the contravariant power set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Set}$ is represented by $\{0,1\}$, because $\mathbf{Set}(A, \{0,1\}) \cong \mathcal{P}A$: a function $f : A \to \{0,1\}$ determines a subset $f^{-1}(1) \subset A$, and a subset $A' \subset A$ defines the indicator map $\mathbf{1}_{A'} : A \to \{0,1\}$ which maps the whole subset A' to 1, and its complement A'^c to 0;

(ii) the Sierpinski space $S := \{\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\}\}$ represents $\mathcal{T} : \mathbf{Top} \to \mathbf{Set}$: given a topological space $(X, \mathcal{T}_X) \in \mathbf{Top}$, an element of the topology $U \in \mathcal{T}_X$ (i.e. an open subset of X) corresponds to the continuous map $\mathbf{1}_U : (X, \mathcal{T}_X) \to S$ (note that it is continuous since $\{1\} \in \mathcal{T}_S$);

(iii) by definition, k represents $U(-)^* = \mathbf{Vect}_k(-, k)$, where

$$U(-)^* : \mathbf{Vect}_k^{op} \xrightarrow{(-)^*} \mathbf{Vect}_k \xrightarrow{U} \mathbf{Set}.$$

Notation and Terminology 2.50. For a locally small category $\mathcal{C}$, we can apply currying (see Example 2.38) to the two-sided represented functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ to obtain two functors

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\text{よ}^\dagger} & \mathbf{Set}^{\mathcal{C}} \\ c & \longmapsto & \mathcal{C}(c,-) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^*} \\ d & \longmapsto & \mathcal{C}(d,-) \end{array}$$

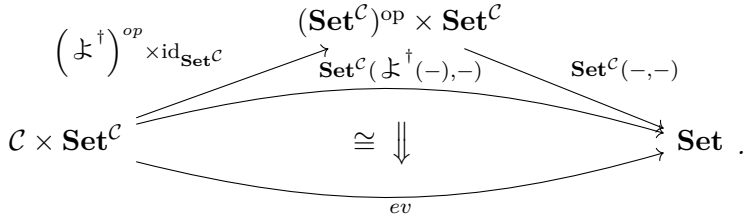
and

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{よ}} & \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \\ c & \longmapsto & \mathcal{C}(-,c) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ d & \longmapsto & \mathcal{C}(-,d) \end{array} .$$

Also, note that there is a functor $ev : \mathcal{C} \times \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$, called *evaluation*, that carries a pair (c, F) to $Fc \in \mathbf{Set}$

The following lemma, called the Yoneda lemma, is one of the most important results in category theory.

Lemma 2.51 (Yoneda lemma)**.** *For any locally small category $\mathcal{C}$, there is a natural isomorphism between the displayed functors:*

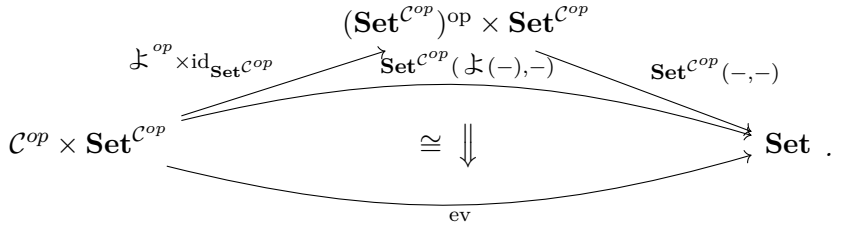


Componentwise: for each object $c \in \mathcal{C}$ and functor $F : \mathcal{C} \to \mathbf{Set}$, there is a bijection

$$\mathbf{Set}^{\mathcal{C}}(C(c,-), F) \xrightarrow{\cong} Fc,$$

which is natural in both variables. A natural transformation $\alpha : \mathcal{C}(c,-) \Rightarrow F$ is mapped to the element $\alpha_c(\mathrm{id}_c)$, while an element $x \in Fc$ is mapped to the natural transformation $\psi(x) : \mathcal{C}(c,-) \Rightarrow F$ defined by $\psi(x)_d(f) = Ff(x)$ for $d \in \mathcal{D}$ and $f \in \mathcal{C}(c,d)$.

Dually, there is a natural isomorphism



Componentwise: for each object $c \in \mathcal{C}$ and functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is a bijection

$$\mathbf{Set}^{\mathcal{C}^{op}}(C(-,c), F) \xrightarrow{\cong} Fc,$$

which is natural in both variables. A natural transformation $\alpha : \mathcal{C}(-,c) \Rightarrow F$ is mapped to the element $\alpha_c(\mathrm{id}_c)$, while an element $x \in Fc$ is mapped to the natural transformation $\psi(x) : \mathcal{C}(-,c) \Rightarrow F$ defined by $\psi(x)_d(f) = Ff(x)$ for $d \in \mathcal{D}$ and $f \in \mathcal{C}(d,c)$.

The Yoneda embedding theorem below is a consequence of Yoneda's lemma. It caracterises all natural transformations between represented functors.

Theorem 2.52. *For any locally small category $\mathcal{C}$, functors $\mathcal{C}^{\mathrm{op}} \xrightarrow{よ^{\dagger}} \mathbf{Set}^{\mathcal{C}}$ and $\mathcal{C} \xrightarrow{よ} \mathbf{Set}^{\mathcal{C}^{op}}$ define embeddings : they are fully faithful and injective on objects.*

In particular, any natural transformation $\alpha : \mathcal{C}(c,-) \Rightarrow C(d,-)$ is the precomposition functor f_* for a unique morphism $f : d \to c \in \mathcal{C}$, which can be recovered by $f = \alpha_c(\mathrm{id}_c)$. Dually, any natural transformation $\mathcal{C}(-,c) \Rightarrow C(-,d)$ is the postcomposition functor g^* for a unique morphism $g : c \to d \in \mathcal{C}$, which can be recovered in the same way.

We can now conclude that any two representations are isomorphic in the following sense.

Corollary 2.53. *Suppose $\mathcal{C}$ is locally small and $F : \mathcal{C} \to \mathbf{Set}$. Assume moreover that (c, α) and (d, β) are two representations of F. Then there exists a unique morphism $f : d \to c$, necessarily an isomorphism, that commutes with the representations:*

$$\begin{array}{ccc} \mathcal{C}(c,-) & \xRightarrow[\cong]{f_*} & \mathcal{C}(d,-) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & F & \end{array} .$$

A dual result applies to contravariant represented functors $F : \mathcal{C}^{op} \to \mathbf{Set}$.

Example 2.54. We can use Theorem 2.52 to find all natural transformations between the functors $\mathrm{Ob}, \mathrm{Mor} : \mathbf{Cat} \rightrightarrows \mathbf{Set}$. By Example 2.48 (vi) and Example 2.48 (vii), they are represented by $\mathbf{1}$ and 2 respectively; in other words, $\mathrm{Ob} \cong \mathbf{Cat}(\mathbf{1}, -)$ and $\mathrm{Mor} \cong \mathbf{Cat}(2, -)$. Applying Lemma 2.24 to the two-sided represented functor of the functor category $\mathbf{Set}^{\mathbf{Cat}}$, we have a bijection

$$\mathbf{Set}^{\mathbf{Cat}}(\mathrm{Mor}, \mathrm{Ob}) \cong \mathbf{Set}^{\mathbf{Cat}}(\mathbf{Cat}(2,-), \mathbf{Cat}(\mathbf{1},-)).$$

Now, Theorem 2.52 implies that

$$\begin{aligned} \mathbf{Cat}(\mathbf{1}, 2) &\cong \mathbf{Set}^{\mathbf{Cat}}(\mathbf{Cat}(2,-), \mathbf{Cat}(\mathbf{1},-). \\ 0 &\longmapsto 0^* \\ 1 &\longmapsto 1^* \end{aligned}$$

Doing the same for natural transformations from Ob to Mor and denoting by $! : 2 \to \mathbf{1}$ the unique functor between these categories, we see that $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor} \Rightarrow \mathrm{Ob}$ and $\mathrm{id} : \mathrm{Ob} \Rightarrow \mathrm{Mor}$ are the only natural transformation between the two functors:

$$\begin{array}{ccc} \mathbf{Cat}(\mathbf{1},-) & \overset{0^*}{\underset{1^*}{\overset{!^*}{\rightrightarrows}}} & \mathbf{Cat}(2,-) \\ \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\ \mathrm{Ob} & \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\overset{\mathrm{id}}{\rightrightarrows}}} & \mathrm{Mor} \end{array} .$$

Definition 2.55. Let $\mathcal{C}$ be locally small. An *element* of a co- or contra -variant set-valued functor F is a pair $(c, x \in Fc)$. An element is *universal* if the corresponding natural transformation $\psi(x)$ given by the Yoneda lemma (Lemma 2.51) is a natural isomorphism.

We can assemble all elements into a single category.

Definition 2.56. Let $\mathcal{C}$ be locally small. For a covariant fucntor $F : \mathcal{C} \to \mathbf{Set}$ the *category of elements* $\int F$ of F

- has as objects elements $(c, x \in Fc)$ of F;
- a morphism $f : (c, x) \to (d, y)$ is a morphism $f : c \to d$ in $\mathcal{C}$ so that $Ff(x) = y$;
- the composite of $(c, x) \xrightarrow{f} (d, y) \xrightarrow{g} (e, z)$ is $g \circ f \in \mathcal{C}$;
- the identity at (c, x) is $\mathrm{id}_c \in \mathcal{C}$.

  Functoriality of F implies that the associativity and identity axioms of a category are satisfied.

Dually, for a contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ the *category of elements* $\int F$ of F

- has as objects elements $(c, x \in Fc)$ of F;
- a morphism $f : (c, x) \to (d, y)$ is a morphism $f : c \to d$ in $\mathcal{C}$ so that $Ff(y) = x$;
- the composite of $(c, x) \xrightarrow{f} (d, y) \xrightarrow{g} (e, z)$ is $g \circ f \in \mathcal{C}$;
- the identity at (c, x) is $\mathrm{id}_c \in \mathcal{C}$.

  Again, functoriality of F implies that the associativity and identity axioms of a category are satisfied.

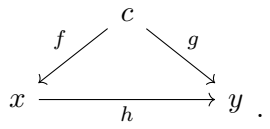
Notation and Terminology 2.57. In order to emphasize the variance, for a functor $F : \mathcal{C} \to \mathbf{Set}$, the category of elements as defined in the covariant case may be denoted by $\int_{\mathcal{C}} F$. Now assume that $F : \mathcal{C}^{op} \to \mathbf{Set}$; the notation $\int^{\mathcal{C}} F$ is used to denote its category of elements according to the dual definition. However, we may also compute the category of elements of F when we view F as a *covariant* functor on $\mathcal{C}^{op}$; in this case, the first definition gives us a category $\int_{\mathcal{C}^{op}} F$. In fact, the dual definition was conceived precisely so that we have the identity

$$\int_{\mathcal{C}^{op}} F \cong \left(\int^{\mathcal{C}} F \right)^{op}.$$

We see that the category of elements can be used to produce new categories from existing functors:

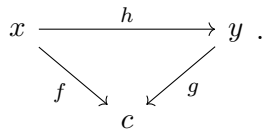
Example 2.58. (i) for a concrete category $\mathcal{C}$ (see Definition 2.41), the category of elements of $U : \mathcal{C} \to \mathbf{Set}$ has as objects "pointed objects" $(c, x \in Uc)$ and as morphisms "pointed maps between the underlying sets" $f : (c, x) \to (c', x')$. For example, taking $U = \mathrm{id}_{\mathbf{Set}}$, we see that $\int U \cong \mathbf{Set}_*$;

(ii) the category of elements $\int \mathcal{C}(c, -)$ has as objects pairs $(x, f : c \to x)$ and a morphism $h : (x, f : c \to x) \to (y, g : c \to y)$ is a morphism $h : x \to y$ so that $g = h \circ f$:



This category is often called the *slice category under* c, denoted by $c/\mathcal{C}$;

(iii) dually, the category of elements $\int \mathcal{C}(-, c)$, referred to as the *slice category over* c, and denoted by $\mathcal{C}/c$, has as objects pairs $(x, f : x \to c)$ and a morphism $h : (x, f : x \to c) \to (y, g : y \to c)$ is a morphism $h : x \to y$ so that $f = g \circ h$:



Unpacking the various definitions, the following result makes a connection between universal elements and initial/terminal objects.

Proposition 2.59. *Let $\mathcal{C}$ be locally small. For a covariant functor $F : \mathcal{C} \to \mathbf{Set}$, an element (c, x) is universal if and only if it is initial in $\int_{\mathcal{C}} F$. In particular, F is representable precisely when its category of elements has an initial object.*

Dually, for a contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, an element (c, x) is universal if and only if it is terminal in $\int^{\mathcal{C}} F$. In particular, F is representable precisely when its category of elements has an terminal object.

It is exactly this initial/terminal object property of a universal element (c, x) in the category of elements that expresses the *universal property* of the pair (c, x).

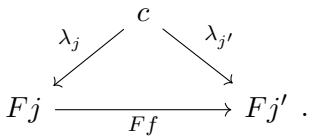
2.3 Limits and Colimits

In this subsection, we use the language introduced in section 2.2 in order to define the key notions of limit and colimit.

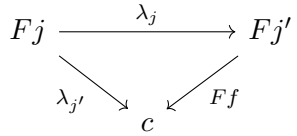
Definition 2.60. For categories $J, \mathcal{C}$, a *J-indexed diagram* in $\mathcal{C}$ is a functor $F : J \to \mathcal{C}$.

Definition 2.61. For categories $J, \mathcal{C}$, for each $c \in C$ there is a *constant functor at c*, $\Delta c : J \to \mathcal{C}$, that sends every object in J to c and every morphism in J to id_c. This construction defines the action on objects of the *constant diagram functor* $\Delta : \mathcal{C} \to \mathcal{C}^J$, that carries a morphism $f : c \to c'$ to the natural transformation $\Delta f : \Delta c \Rightarrow \Delta c'$ whose component at every object in J is f.

Definition 2.62. A *cone* with summit $c \in \mathcal{C}$ over a diagram $F : J \to \mathcal{C}$ is a natural transformation $\lambda : \Delta c \Rightarrow F$. Explicitly, it consists of a family of morphisms $(\lambda_j : c \to Fj)_{j \in J}$ indexed by J, called *legs* of the cone, satisfying the following compatibility condition: for every $f : j \to j' \in J$, the following triangle commutes in $\mathcal{C}$:



Dually, a cone under F (also called *cocone*) with nadir $c \in \mathcal{C}$, is a natural transformation $\lambda : F \Rightarrow \Delta c$; equivalently, it consists of a family of morphisms $(\lambda_j : Fj \to c)_{j \in J}$ so that for all $f : j \to j' \in J$, the triangle



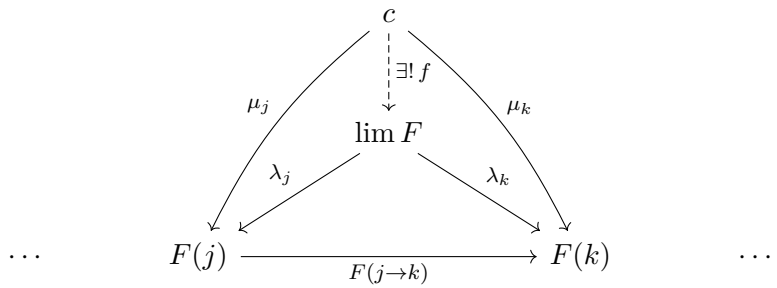
commutes.

Definition 2.63. For any diagram $F : J \to \mathcal{C}$ where J is small and $\mathcal{C}$ is locally small (so that $\mathcal{C}^J$ is locally small), there is a functor

$$\mathcal{C}^J(\Delta -, F) : \mathcal{C}^{op} \xrightarrow{\Delta^{op}} (\mathcal{C}^J)^{op} \xrightarrow{\mathcal{C}^J(-,F)} \mathbf{Set}$$

that sends an object $c \in \mathcal{C}$ to the set of cones with summit c over F, and a morphism $f : c \to d$ to the function $(\Delta f)^* : \mathcal{C}^J(\Delta d, F) \to \mathcal{C}^J(\Delta c, F)$ which converts a cone $\mu : \Delta d \Rightarrow F$ with summit d to the cone $\mu \circ \Delta f : \Delta c \Rightarrow F$ with summit c by precomposing with Δf.

A *limit* of F is a representation for the contravariant set-valued functor $\mathcal{C}^J(\Delta -, F)$; by the Yoneda lemma, it consists of an object $\lim F \in \mathcal{C}$ together with a universal cone $\lambda : \Delta \lim F \Rightarrow F$, called *limit cone*, with the following universal property: for any $c \in \mathcal{C}$ and cone $\mu : \Delta c \Rightarrow F$ with summit c, $\exists!\ f : c \to \lim F$ so that $\mu = \lambda \circ \Delta f$; equivalently, for each $j \in J$, $\mu_j = \lambda_j \circ f$:

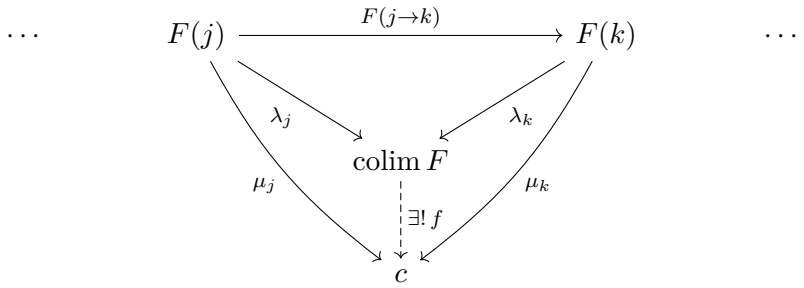


Dually, there is a functor

$$\mathcal{C}^J(F, \Delta -) : \mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J \xrightarrow{\mathcal{C}^J(F,-)} \mathbf{Set}$$

that sends an object $c \in \mathcal{C}$ to the set of cocones with nadir c under F, and a morphism $f : c \to d$ to the function $(\Delta f)_* : \mathcal{C}^J(F, \Delta c) \to \mathcal{C}^J(F, \Delta d)$ which converts a cone $\mu : F \Rightarrow \Delta c$ with nadir c to the cone $\Delta f \circ \mu : F \Rightarrow \Delta d$ with nadir d by postcomposing with Δf.

A *colimit* of F is a representation for the covariant set-valued functor $\mathcal{C}^J(F, \Delta -)$; by the Yoneda lemma, it consists of an object $\operatorname{colim} F \in \mathcal{C}$ together with a universal cocone $\lambda : F \Rightarrow \Delta \operatorname{colim} F$, called *colimit cone*, with the following universal property: for any $c \in \mathcal{C}$ and cocone $\mu : F \Rightarrow \Delta c$ with nadir c, $\exists!\ f : \operatorname{colim} F \to c$ so that $\mu = \Delta f \circ \lambda$; equivalently, for each $j \in J$, $\mu_j = f \circ \lambda_j$:



Remark 2.64. Note that the duality between cones and cocones can be expressed precisely in the following way:

$$\left(\int_{\mathcal{C}} \mathcal{C}^J(F, \Delta -)\right)^{op} \cong \int^{\mathcal{C}^{op}} (C^{op})^{J^{op}}(\Delta -, F^{op}) \,,$$

where on the left the category of elements is computed with respect to the covariant functor on $\mathcal{C}$, and on the right it is computed with respect to the contravariant functor on $\mathcal{C}^{op}$. In words, it expresses that cocones under a diagram $F : J \to \mathcal{C}$ correspond to cones over the diagram $F^{op} : J^{op} \to \mathcal{C}^{op}$ (with the additional property that morphisms of cocones correspond to morphisms of cones with swapped domain and codomain).

The category of elements $\int^{\mathcal{C}} \mathcal{C}^J(\Delta -, F)$ is also referred to as the category of cones over F; dually, the *category of elements* $\int_{\mathcal{C}} \mathcal{C}^J(F, \Delta -)$ is called *category of cocones* under F.

Proposition 2.65. *For any two limit cones λ, λ', there is a unique isomorphism f between the summits commuting with the legs of the limit cones; that is, $\lambda = \lambda' \circ \Delta f$, i.e. $\lambda'_j = \lambda_j \circ f$ for all $j \in J$.*

The dual result describes the essential uniqueness of colimit cones.

We now present examples of limits and colimits. We will describe both dual notions in detail.

Example 2.66. The limit of an empty diagram, indexed by the empty category, is a terminal object in $\mathcal{C}$. Dually, a colimit in this case is an initial object in $\mathcal{C}$.

Example 2.67 (Products and coproducts)**.** A *product* is a limit of a diagram $F : J \to \mathcal{C}$ indexed by a discrete category. Explicitly, it is an object $\prod_{j \in J} Fj$ together with so-called *product projections* $(\pi_k : \prod_{j \in J} Fj \to Fk)_{k \in J}$ so that for every object $X \in \mathcal{C}$ and maps $(f_j : X \to Fj)_{j \in J}$, there is a unique map $f : X \to \prod_{j \in J} Fj$ so that $f_j = \pi_j \circ f$ for all $j \in J$. In particular, to specify a map to the product, it suffices to specify the components $(f_j)_{j \in J}$.

Dually, a *coproduct* is a colimit of a diagram $F : J \to \mathcal{C}$ indexed by a discrete category. It consists of an object $\coprod_{j \in J} Fj$ together with so-called *coproduct inclusions* $(\iota_k : Fk \to \coprod_{j \in J} Fj)_{k \in J}$ so that for every object $X \in \mathcal{C}$ and maps $(f_j : Fj \to X)_{j \in J}$, there is a unique map $f : \coprod_{j \in J} Fj \to X$ so that $f_j = f \circ \iota_j$ for all $j \in J$. In particular, to specify a map from the coproduct, it suffices to specify the components $(f_j)_{j \in J}$.

We describe examples of products and coproducts in familiar categories.

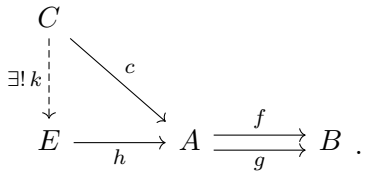
Example 2.68. (i) In **Set**, the categorical product of a family of sets $(Ai)_{i\in I}$ is the usual cartesian product $\prod_{i\in I} A_i$, which explains the notation in Example 2.67. The categorical coproduct is the disjoint union of sets.

(ii) In **Top**, the product of the spaces is the cartesian product of the underlying sets, equipped with the product topology. The coproduct of spaces is the disjoint union of the underlying sets, whose open sets are exactly disjoint unions of open sets in each space.

(iii) In **Group**, the product is the usual direct product of groups, where the group law is defined componentwise. The coproduct of groups is more difficult to describe and will be discussed in Proposition 2.97 and Theorem 3.52 .

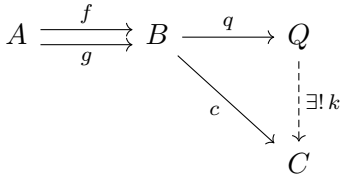
(iv) In $\mathbf{Vect}_k$, the product is the usual direct product of vector spaces, where vector addition and scalar multiplication are again defined componentwise. The coproduct of vector spaces $(V_i)_{i\in I}$ is the direct sum $\bigoplus_{i\in I} V_i$, whose elements are sequences $(v_i \in V_i)_{i\in I}$ where $v_i \neq 0$ for only finitely many $i \in I$; addition and scalar multiplication are again defined componentwise. Note that if I is finite, the notions of product and coproduct of vector spaces coincide.

Example 2.69 (Equalizers and coequalizers)**.** An *equalizer* is a limit of a diagram indexed by the parallel pair $\bullet \rightrightarrows \bullet$. Thus a diagram is given by a parallel pair of morphisms $f, g : A \rightrightarrows B$; the data of a limit is an object E together with a morphism $h : E \to A$ so that $f \circ h = g \circ h$ satisfying the following universal property: given any other morphism $c : C \to A$ so that $f \circ c = g \circ c$, there is a unique $k : C \to E$ so that $c = h \circ k$:



An equalizer $E \overset{h}{\rightarrowtail} A$ is automatically a monomorphism.

Dually, a *coequalizer* is a colimit of a diagram indexed by the parallel pair $\bullet \rightrightarrows \bullet$. Given a diagram $f, g : A \rightrightarrows B$, the data of a colimit is an object Q together with a morphism $q : B \to Q$ so that $q \circ f = q \circ g$ satisfying the following universal property: given any other morphism $c : B \to C$ so that $c \circ f = c \circ g$, there is a unique $k : Q \to C$ making the bottom-right triangle commute:



Note that a coequalizer $B \overset{q}{\twoheadrightarrow} Q$ is an epimorphism.

These concepts take the following form in familiar categories.

Example 2.70. (i) In **Set**, the categorical equalizer of a parallel pair $f, g : X \rightrightarrows Y$ is $\{x \in X \mid f(x) = g(x)\} \subset X$ together with its inclusion into X - which illustrates the general property that equalizers are monomorphisms. The coequalizer is $Y/\sim$, where $\sim$ is the smallest equivalence relation which identifies $f(x) \sim g(x)$ for each $x \in X$, together with the quotient map - which is indeed an epimorphism as expected.

(ii) In **Top**, the formulas for the underlying sets of the equalizers and coequalizers are the same as the equalizers and coequalizers of the underlying sets in **Set**; the equalizer is equipped with the subspace topology, and the coequalizer with the quotient topology.

(iii) In **Group**, the equalizer of a parallel pair of group homomorphisms $\phi, \psi : G \rightrightarrows H$ is the subgroup $\{\mathrm{g} \in G \mid \phi(\mathrm{g}) = \psi(\mathrm{g})\} \leq G$; the coequalizer is the quotient H/N, where $N := \langle\langle \phi(g)\psi(g)^{-1} \mid g \in G \rangle\rangle \trianglelefteq H$, or equivalently, the smallest normal subgroup so that for all $g \in G$, $\phi(g)$ and $\psi(g)$ are identified in the quotient of H by the normal subgroup.

(iv) In $\mathbf{Vect}_k$, the equalizer of a parallel pair of linear maps $\phi, \psi : V \rightrightarrows W$ is the subspace $\ker(\phi - \psi)$; the coequalizer is $W/\operatorname{im}(\phi - \psi)$.

The lemma below is used frequently to determine whether maps into a limit object or out of a colimit object are equal.

Lemma 2.71. *Let $F : J \to \mathcal{C}$ be a diagram, with limit cone $\lambda : \Delta \lim F \Rightarrow F$; and $f, g : X \rightrightarrows \lim F$ a parallel pair whose codomain is the limit object. Then $f = g$ if and only if $\lambda_j \circ f = \lambda_j \circ g$ for all $j \in J$: it suffices to check equality after postcomposing with the legs of the limit cone.*

Dually, if $\lambda : F \Rightarrow \Delta \operatorname{colim} F$ is a colimit cone, and $f, g : \operatorname{colim} F \rightrightarrows X$ a parallel pair whose domain is the colimit object, then $f = g$ if and only if $f \circ \lambda_j = g \circ \lambda_j$ for all $j \in J$: it suffices to check equality after precomposing with the legs of the colimit cone.

Proposition 2.72. *Let $\mathcal{C}$ be a category which admits limits of any J-shaped diagram. Then the choice of a limit cone for any such diagram defines a functor $\lim : \mathcal{C}^J \to \mathcal{C}$.*

Remark 2.73. Explicitly, a J-indexed diagram $F : J \to \mathcal{C}$ with chosen limit cone $\lambda : \lim F \Rightarrow F$ is mapped to $\lim F$; if $G : J \to \mathcal{C}$ has chosen limit cone $\mu : \Delta \lim G \Rightarrow G$, then a natural transformation $\alpha : F \Rightarrow G$ induces a cone $\alpha \circ \lambda : \Delta \lim F \Rightarrow G$, which by the universal property of the limit cone μ gives rise to a unique morphism $\lim \alpha : \lim F \to \lim G$ between the limit objects so that $\alpha \circ \lambda = \mu \circ \Delta \lim \alpha$.

A dual result describes functoriality of colimits.

Example 2.74. On account of the isomorphism $\mathcal{C} \times \mathcal{C} \cong \mathcal{C}^{\bullet\,\bullet}$, any category $\mathcal{C}$ which admits binary products (i.e. limits of any diagram indexed by the discrete category on two objects $\bullet\ \bullet$) has, by Proposition 2.72, a product functor

$$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

It maps a pair of objects $X, Y \in \mathcal{C}$ to a chosen product binary product $X \times Y$, and a pair of morphisms $f : X \to X'$, $g : Y \to Y'$ to the morphism $f \times g : X \times X' \to Y \times Y'$, whose components are specified by

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \uparrow{\scriptstyle \pi_X} & & \uparrow{\scriptstyle \pi_Y} \\ X \times X' & \xrightarrow{\quad f \times g \quad} & Y \times Y' \\ \downarrow{\scriptstyle \pi_{X'}} & & \downarrow{\scriptstyle \pi_{Y'}} \\ X' & \xrightarrow{\quad g \quad} & Y' \end{array} .$$

Proposition 2.75. *Let $\mathcal{C}$ be a category which admits limits of any J-shaped diagram. For any two limit functors $\lim, \lim' : \mathcal{C}^J \to \mathcal{C}$, there is a natural isomorphism $\lim \overset{\cong}{\Longrightarrow} \lim'$ whose components are defined by Proposition 2.65.*

To motivate the following definitions, consider a functor $F : \mathcal{C} \to \mathcal{D}$, and a J-indexed diagram $K : J \to \mathcal{C}$ valued in $\mathcal{C}$. If K admits a limit cone $\lambda : \Delta \lim K \Rightarrow K$ in $\mathcal{C}$, it need not be the case that its image $F\lambda : \Delta F \lim K = F\Delta \lim K \Rightarrow FK$ under F defines a limit cone over FK in $\mathcal{D}$. However, if FK admits a limit cone μ, its universal property would imply the existence of a comparison map

$$\kappa : F \lim K \to \lim FK \tag{2.2}$$

so that $F\lambda = \mu \circ \Delta\kappa$.

Definition 2.76. Given any diagram $K : J \to \mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$

- *preserves* limits of K if the image under F of any limit cone over K defines a limit cone over FK;
- *reflects* limits of K if any cone over K, whose image under F is a limit cone over FK, defines a limit cone over K;
- *creates* limits of K if whenever FK has a limit, then K has a limit; and in this case, we require preservation and reflection of limits of K: a cone over K defines a limit cone if and only if its image under F is a limit cone over FK;
- *strictly creates* limits of K if any limit cone over FK admits a unique lift to a cone over K, which is required to be a limit cone over K.

More generally, we may use these three terms relative to a class of diagrams; for instance, a functor F preserves limits of a given class of diagrams if it preserves limits of any diagram which belongs to that class.

There is a dual definition for colimits.

Remark 2.77. Unsurprisingly, strict creation is a stronger property than creation.

Lemma 2.78. *A fully faithful functor reflects limits and colimits.*

If in addition the functor is essentially surjective on objects (i.e. it defines an equivalence by Theorem 2.43), we obtain two additional properties.

Lemma 2.79. *Any equivalence of categories preserves, reflects and creates limits and colimits.*

If we desire a stronger property regarding limits, we need a stronger property of the functor:

Lemma 2.80. *Any isomorphism of categories strictly creates limits.*

In Definition 3.35, we will encounter another notion (here, that of limits/colimits) which highlights the distinction between an isomorphism and an equivalence of categories: the former "strictly creates" certain mathematical objects, while the latter merely "creates" them.

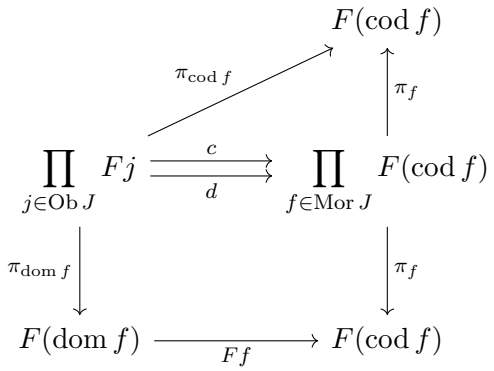
Definition 2.81. A diagram is *small* if its indexing category is small. A category $\mathcal{C}$ is *complete* if it admits limits of any small diagram valued in $\mathcal{C}$, and cocomplete if it has all small colimits valued in $\mathcal{C}$.

The following theorem asserts that the mere existence of products and equalizers automatically implies that any small diagram, no matter its shape, admits a limit.

Theorem 2.82. *Let $F : J \to \mathcal{C}$ be a diagram. Assuming that the specified products and equalizer exist, then $\lim_J F$ exists and can be computed in the following way:*

$$\lim_J F \rightarrowtail \prod_{j \in \mathrm{Ob}\, J} Fj \overset{\overset{c}{\longrightarrow}}{\underset{d}{\longrightarrow}} \prod_{f \in \mathrm{Mor}\, J} F(\mathrm{cod}\, f)\ ,$$

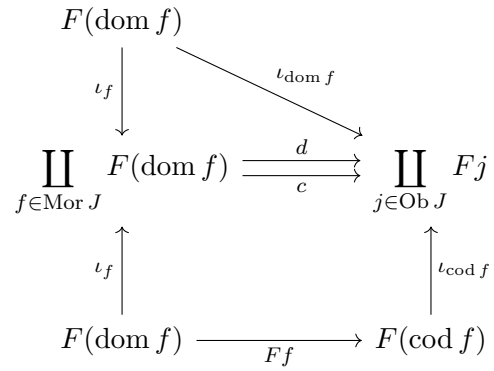
where the components of the maps c, d are given by



Dually, if the specified coproducs and coequalizers exist, then F admits a colimit, computed as follows:

$$\coprod_{f\in \operatorname{Mor} J} F(\operatorname{dom} f) \underset{c}{\overset{d}{\rightrightarrows}} \coprod_{j\in \operatorname{Ob} J} Fj \twoheadrightarrow \operatorname*{colim}_J F\,,$$

where the components of c, d are defined by

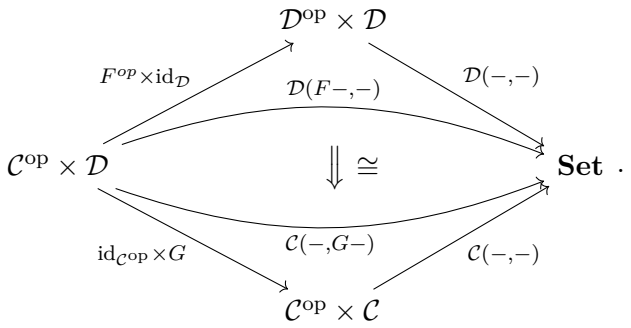


Example 2.83. Now Theorem 2.82, Example 2.68 and Example 2.70 imply that **Set**, **Top**, **Group** and $\mathbf{Vect}_k$ are complete and cocomplete.

2.4 Adjunctions

Adjunctions are one of the central concepts of category theory.

Definition 2.84. An *adjunction* consists of a pair of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ pointing in opposite directions, together with a natural isomorphism $\mathcal{D}(F-,-) \overset{\cong}{\Longrightarrow} \mathcal{C}(-,G-)$, as displayed below:



Explicitly, the naturality condition states that for any $h : c' \to c \in \mathcal{C}$ and $k : d \to d' \in \mathcal{D}$, the outer square commutes:

$$\begin{array}{ccc} \mathcal{D}(Fc, d) & \xrightarrow{\cong} & \mathcal{C}(c, Gd) \\ {\scriptstyle (Fh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathcal{D}(Fc', d) & & \mathcal{C}(c', Gd) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle (Gk)_*} \\ \mathcal{D}(Fc', d') & \xrightarrow{\cong} & \mathcal{C}(c', Gd') \end{array} .$$

Notation and Terminology 2.85. An adjunction is often denoted by

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$$

or more concisely by $F \dashv G$.

Notation and Terminology 2.86. Under the bijection $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$, corresponding pairs, called *transposes* of each other, are denoted by

$$\begin{array}{ccc} \mathcal{D}(Fc, d) & \xleftrightarrow{\cong} & \mathcal{C}(c, Gd) \\ f & \longmapsto & f^\sharp \\ g^\flat & \longmapsfrom & g \end{array} .$$

In particular, the relations

$$(f^\sharp)^\flat = f$$

and

$$(g^\flat)^\sharp = g$$

hold.

The following lemma gives another point of view on the naturality condition.

Lemma 2.87. *Consider functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ so that we have isomorphisms $D(Fc, d) \cong C(c, Gd)$ $\forall c \in \mathcal{C}, \forall d \in \mathcal{D}$. Then naturality in both varibles of these isomorphisms is equivalent to the condition that one square below commutes if and only if the other commutes:*

$$\begin{array}{ccc} Fc & \xrightarrow{f^\sharp} & d \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow[g^\sharp]{} & d' \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow[g^\flat]{} & Gd' \end{array} .$$

The examples below demonstrate that adjunctions are ubiquitous.

Example 2.88. (i) The *free group* construction from group theory defines a functor $F :$ **Set** $\to$ **Group** which provides a left adjoint to the forgetful functor $U :$ **Group** $\to$ **Set**. Explicitly, F carries a set X to the free group FX on X consisting of all formal finite words ${x_1}^{\pm 1}{x_2}^{\pm 1} \cdots {x_n}^{\pm 1}$ made up of elements of X and formal inverses thereof (there is a rigorous construction which we will not present here as it is not trivial to define); it carries a function $f : X \to Y$ to the group homomorphism $Ff : FX \to FY$ which replaces each occurrence of an element $x \in X$ in a finite formal word by $f(x) \in Y$. To see that there is

an adjunction $F \dashv U$, note that for any set $X \in \mathbf{Set}$ and group $G \in \mathbf{Group}$, there is a bijection

$$\mathbf{Group}(FX, G) \cong \mathbf{Set}(X, UG).$$

It carries a group homomorphism $\phi : FX \to G$ to the map $\phi\flat : \mathrm{x} \in X \mapsto \phi(\mathrm{x}) \in UG$ (where we view an element $\mathrm{x} \in X$ as a one-letter word in FX), and a function $h : X \to UG$ to $h^\sharp : \mathrm{x_1}^{\pm 1}\mathrm{x_2}^{\pm 1} \cdots \mathrm{x_n}^{\pm 1} \in FX \mapsto \mathrm{h(x_1)}^{\pm 1}\mathrm{h(x_2)}^{\pm 1} \cdots \mathrm{h(x_n)}^{\pm 1} \in G$ (where the right-hand product is computed in G). Notice in particular that to define this bijection, we made use of an "inclusion" map from a set to the free group on that set, as well as an "evaluation" map from the free group on (the underlying set of) the group to the group with computes the formal finite word inside the group. This is not a coincidence: they define the very notion of an adjunction by Theorem 2.89.

(ii) There are $D, I : \mathbf{Set} \to \mathbf{Top}$ that carry a set S to the *discrete* $DS = (S, \mathcal{P}S)$ and *indiscrete* $IS = (S, \{\emptyset, S\})$ topologies on S respectively. Then we have an *adjoint triple*

$$D \dashv U \dashv I,$$

where $U : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor. Indeed, for any set S and space $(X, \mathcal{T})$, we have bijections

$$\mathbf{Top}(DS, (X, \mathcal{T})) \cong \mathbf{Set}(S, U(X, \mathcal{T}) = X)$$

and

$$\mathbf{Set}(U(X, \mathcal{T}) = X, S) \cong \mathbf{Top}((X, \mathcal{T}), IS)\ ,$$

because any map with discrete domain or indiscrete codomain is automatically continuous.

(iii) For preorders $(A, \leq), (B, \leq)$ and functors (equivalently, order-preserving maps by Example 2.19 (xi)) $F : (A, \leq) \to (B, \leq)$, $G : (B, \leq) \to (A, \leq)$, there is an adjunction $F \dashv G$ if and only if

$$\forall a \in A,\ \forall b \in B,\ Fa \leq b \iff a \leq Gb.$$

To illustrate, note that any function $f : X \to Y$ induces order-preserving maps $f_* : \mathcal{P}X \to \mathcal{P}Y$ and $f^{-1} : \mathcal{P}Y \to \mathcal{P}X$; since for any subsets $A \subset X,\ B \subset Y$:

$$f(A) \subset B \iff A \subset f^{-1}(B)\ ,$$

there is an adjunction $f_* \dashv f^{-1}$.

(iv) The forgetful functor $U : \mathbf{Set}_* \to \mathbf{Set}$ has a left adjoint $- \sqcup \{-\} : \mathbf{Set} \to \mathbf{Set}_*$ that carries a set X to the pointed set $(X \sqcup \{X\}, X)$ (we choose the freely adjoined point to be the set X itself, being the only natural choice given a set X) and a function $f : X \to Y$ to the corresponding pointed map, which sends the new point $X \in X \sqcup \{X\}$ to $Y \in Y \sqcup \{Y\}$. The adjunction comes from the evident bijection

$$\mathbf{Set}_*((X \sqcup \{X\}, X), (Y, \mathrm{y})) \cong \mathbf{Set}(X, Y)$$

for any set $X \in \mathbf{Set}$ and pointed set $(Y, \mathrm{y}) \in \mathbf{Set}_*$.

The following theorem gives an equivalent definition of an adjunction.

Theorem 2.89. *The data of an adjunction $F \dashv G$ is equivalent to the data of a natural transformation $\eta : \mathrm{id}_\mathcal{C} \Rightarrow GF$, called unit, and a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_\mathcal{D}$, called counit, satisfying the following* triangle identities*:*

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FGF \\ & \searrow^{\mathrm{id}_F} & \Downarrow \varepsilon F \\ & & F \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} G & \xRightarrow{\eta G} & GFG \\ & \searrow^{\mathrm{id}_G} & \Downarrow G\varepsilon \\ & & G \end{array}\ .$$

The correspondance is defined as follows. From an adjunction $F \dashv G$ with natural isomorphisms $D(Fc, d) \cong C(c, Gd)$, the component of the unit at $c \in \mathcal{C}$ is defined to be $\eta_c := \mathrm{id}_{Fc}^\flat : c \to GFc$, the transpose of the identity at Fc; the component of the counit at $d \in \mathcal{D}$ is $\varepsilon_d := \mathrm{id}_{Gd}^\sharp$, the transpose of the identity at Gd.

Conversely, given η and ε satisfying the triangle identities, the hom-set bijections are defined by

$$\mathcal{D}(Fc, d) \xrightarrow{G} \mathcal{C}(GFc, Gd) \xrightarrow{(\eta_c)^*} \mathcal{C}(c, Gd)$$

$$Fc \xrightarrow{f^\sharp} d \longmapsto f^\flat := c \xrightarrow{\eta_c} GFc \xrightarrow{Gf^\sharp} Gd$$

with inverse

$$\mathcal{D}(Fc, d) \xleftarrow{(\varepsilon_d)_*} \mathcal{D}(Fc, FGd) \xleftarrow{F} \mathcal{C}(c, Gd)$$

$$g^\sharp := Fc \xrightarrow{Fg^\flat} FGd \xrightarrow{\varepsilon_d} d \longleftarrow c \xrightarrow{g^\flat} Gd \,.$$

We now illustrate the concept of adjunction thanks to this equivalent formulation, while also revisiting some adjunctions from Example 2.88 .

Example 2.90. (i) We describe the unit and counit of the free-forgetful adjunction for groups, using the forward direction of the bijection from Example 2.88 (i). For a set S, the transpose of id_{FS} is $\eta_S := \mathrm{id}_{FS}^\sharp : S \to UFS$, which sends an element $s \in S$ the one-letter word $s \in UFS$. For a group G, the transpose of id_{UG} is the unique group homomorphism $\varepsilon_G : FUG \to G$ so that

$$UG \xrightarrow{\eta_{UG}} UFUG \xrightarrow{U\varepsilon_G} UG$$

is the identity. In particular, we discover that for a one-letter word $g \in FUG$, it holds that $\varepsilon_G(g) = g$; the fact that ε_G is a group homomorphism implies that any finite formal words made up of elements of G and formal inverses is evaluated in G using the familiar inversion and multiplication inside the group G.

By Theorem 2.89, we already know that η and ε are natural and satisfy the triangle identities; but if we prefer not to give an explicit bijection as done in Example 2.88 and prove naturality in two variables, it is also possible to start from η and ε as defined above, prove that they are natural and satisfy the triangle identities directly, in order to conclude that we have an adjunction $F \dashv U$, which induces natural bijections in both variables.

(ii) We present a construction of the *free monoid* on a set. Consider a set X and choose a symbol e not belonging to X; define

$$\overline{X} := \{ s : \mathbb{N} \to X \sqcup e \mid \exists n \in \mathbb{N} \; s.t. \; s_k \in X \; \forall k < n, \; s_k = e \; \forall k \geq n \}$$

to be the set of sequences which eventually end in the chosen symbol e - in other words, finite sequences in X with arbitrary length. We also denote by

$$\eta_X^{(n)} : (\mathrm{x}_1, \cdots, \mathrm{x}_n) \in X^n \mapsto (\mathrm{x}_1, \cdots, \mathrm{x}_n, e \cdots) \in \bar{X}$$

the inclusion map. The set $\bar{X}$ can be equipped with a monoid structure $FX = (\bar{X}, *, 1)$, where the monoid law is concatenation (we glue the "X" parts of two sequences together and extend it with e's) which is clearly associative; it admits a unital identity $1 := (e, e, \cdots)$.

To define the action on morphisms, a function $f : X \to Y$ is mapped to the monoid homomorphism

$$Ff : (\mathrm{x}_1, \cdots, \mathrm{x}_n, e \cdots) \in FX \mapsto (f(\mathrm{x}_1), \cdots, f(\mathrm{x}_n), e' \cdots).$$

This defines a functor $F : \mathbf{Set} \to \mathbf{Monoid}$. For a set X, define $\eta_X := \eta^1_X : X \to UFX$ and for a monoid $M = (M, \star, u)$,

$$\varepsilon_M : (\mathrm{m}_1, \cdots, \mathrm{m}_n, e \cdots) \in FUM \mapsto \mathrm{m}_1 \star \cdots \star \mathrm{m}_n \in M \ ,$$

which evaluates the formal finite word inside M using the monoid law $\star$. After checking naturality of η and ε, it remains to verify the triangle identities

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FUFX \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \varepsilon_{FX}} \\ & & FX \end{array} \qquad \text{and} \qquad \begin{array}{ccc} UM & \xrightarrow{\eta_{UM}} & UFUM \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle U\varepsilon_M} \\ & & UM \end{array}$$

for any set $X \in \mathbf{Set}$ and monoid $M \in \mathbf{Monoid}$, which follow from the definitions. We thus obtain an adjunction $F \dashv U$.

(iii) Given a ring R, we aim to construct the *free left R-module* on a set, which will provide a left adjoint to the forgetful functor $U : {}_R\mathbf{Mod} \to \mathbf{Set}$. For a set S, define

$$R[S] := \{\chi : S \to R \mid \chi(s) \neq 0 \text{ for only finitely many } s \in S\},$$

augmented to an R-module with addition and scalar multiplication by $r \in R$ defined pointwise in R. An element $\chi \in R[S]$ is often denoted by a "formal finite linear combination" $\sum_{s \in S} \chi(s) \cdot s$, which makes it easier to visualize addition and scalar multiplication, by "grouping" the coefficients in front of an element $s \in S$ and "distributing" an element $r \in R$ respectively.

To define the action on morphisms, a function $f : S \to T$ defines an R-linear map $R[f] : R[S] \to R[T]$ which carries a finitely supported function $\chi \in R[S]$ to

$$R[f](\chi) : t \in T \mapsto \sum_{s \in S : f(s) = t} \chi(s) \in R;$$

informally:

$$R[f](\chi) = \sum_{s \in S} \chi(s) \cdot f(s) \in R[T].$$

We thus obtain a functor $R[-] : \mathbf{Set} \to {}_R\mathbf{Mod}$. Moreover, the unit, whose component at a set S is $\eta_S : s \in S \to \mathbf{1}_s \in UR[S]$ (where $\mathbf{1}_s(s') = 1$ if s' = s and $\mathbf{1}_s(s') = 0$ if s' $\neq$ s), and the counit, whose component at a left R-module M is defined by the evaluation

$$\varepsilon_M : \chi \in FUM \to \sum_{m \in M} \chi(m) \cdot m \in M,$$

define an adjunction $R[-] \dashv U$.

If we choose the ring R to be a field k, we see that the *free k-vector space* $k[-]$ provides a left adjoint to $U : \mathbf{Vect}_k \to \mathbf{Set}$. Moreover, since an abelian group is a $\mathbb{Z}$-module, the choice $R = \mathbb{Z}$ yields the *free abelian group* functor $\mathbb{Z}[-]$, left adjoint to $U : \mathbf{Ab} \to \mathbf{Set}$.

(iv) The unit and counit of the adjunction from Example 2.88 (iv) are given by the inclusion $\eta_X : X \to X \sqcup \{X\}$ for a set X; and $\varepsilon_{(Y,y)} : (Y \sqcup \{Y\}, Y) \to (Y, y)$ for a pointed set (Y, y), which is the identity on Y, and sends the freely-adjoined point $Y \in Y \sqcup \{Y\}$ to the point $y \in Y$.

Proposition 2.91. *Given adjunctions*

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D}$$

with unit η and counit ε, and

$$\mathcal{C}' \underset{G'}{\overset{F'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}'$$

with unit η' and counit ε', as well as a pair of functors $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ commuting with the left and right adjoints (that is, we have $KF = F'H$ and $HG = G'K$)

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ {\scriptstyle F}\uparrow \dashv \downarrow{\scriptstyle G} & & {\scriptstyle F'}\uparrow \dashv \downarrow{\scriptstyle G'} \\ \mathcal{C}' & \xrightarrow{H} & \mathcal{C} \end{array}$$

the following conditions are equivalent.

(i) $H\eta = \eta' H$

(ii) $K\varepsilon = \varepsilon' K$

(iii) Transposition across the adjunctions commutes with the applications of K and H; that is, for any $c \in \mathcal{C}$, $d \in \mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{D}(Fc, d) & \xrightarrow{K} & \mathcal{D}'(KFc, Kd) & = & \mathcal{D}'(F'Hc, Kd) \\ {\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle\cong} \\ \mathcal{C}(c, Gd) & \xrightarrow[H]{} & \mathcal{C}'(Hc, HGd) & = & \mathcal{C}'(Hc, G'Kd) \end{array} .$$

Notation and Terminology 2.92. In the setting of Proposition 2.91, we say that (H, K) is a *strict morphism of adjunctions*. If one of the functors is an identity, we may, by abuse of terminology, refer to the other functor as a *strict morphism of adjunctions*.

The proposition below sates that when a functor admits a left adjoint, it is unique (up to isomorphism).

Proposition 2.93. *If a functor $G : \mathcal{D} \to \mathcal{C}$ admits two left adjoints $F, F' : \mathcal{C} \to \mathcal{D}$, then F and F' are naturally isomorphic via $F \overset{F\eta'}{\Longrightarrow} FGF' \overset{\varepsilon F'}{\Longrightarrow} F'$.*

The dual result describes uniqueness of right adjoints.

The following theorem provides a crucial connection between section section 2.3 and section section 2.4.

Theorem 2.94. *Right adjoints preserve limits, and left adjoints preserve colimits.*

Example 2.95. For any two sets $S, T \in$ **Set**, since the left adjoint $F :$ **Set** $\to$ **Group** from Example 2.88 (i) preserves colimits (and hence coproducts) by Theorem 2.94, we obtain

$$F(S \sqcup T) \cong FS * FT,$$

where $*$ denotes the free product - the categorical coproduct - of two groups.

2.5 Applications of Theorem 2.94

This subsection, due to the present author, explores some applications of Theorem 2.94.

The following result provides a connection between concepts from sections 2.1, 2.3 and 2.4.

Proposition 2.96. *Let $U : \mathcal{D} \to \mathbf{Set}$ be a $\mathbf{Set}$-valued functor. Then U admits a left adjoint if and only if*

- *U is represented by some object $d \in \mathcal{D}$;*
- *$\mathcal{D}$ admits arbitrary coproducts of the representing object $d \in \mathcal{D}$.*

Proof. $\Rightarrow$: Assume U admits a left adjoint $F : \mathbf{Set} \to \mathcal{D}$. In particlar, there is a natural isomorphism

$$\mathcal{D}(Fc, D) \cong \mathbf{Set}(c, UD).$$

Choosing $c = *$ (the singleton set), we obtain an isomorphism by Example 2.48 (ii)

$$\mathcal{D}(F*, D) \cong \mathbf{Set}(*, UD) \cong UD,$$

naturally in $D \in \mathcal{D}$. In particular, $d := F*$ represents U. Moreover, since by Theorem 2.94 left adjoints preserve coproducts, we see that for any set X:

$$FX = F(\coprod_{x \in X} \{x\}) \cong \coprod_{x \in X} F(\{x\}) \cong \coprod_{x \in X} d,$$

which proves that $\mathcal{D}$ admits arbitrary coproducts of d.

$\Leftarrow$: Conversly, assume that $d \in \mathcal{D}$ represents U and $\mathcal{D}$ admits arbitrary coproducts of d. We define a functor $F : \mathbf{Set} \to \mathcal{D}$. For $X \in \mathbf{Set}$, we set

$$FX := \coprod_{x \in X} d$$

and for a function $f : X \to Y$, the components of the morphism Ff are defined by

$$\begin{array}{ccc} & d & \\ & \downarrow{\scriptstyle i_x} \quad \searrow^{i_{f(x)}} & \\ FX = \coprod_{x \in X} d & \xrightarrow[Ff]{} & \coprod_{y \in Y} d = FY \end{array} .$$

Using the universal property of the coproduct as well as the fact that d represents U, we obtain a natural isomorphism

$$\mathcal{D}(FX, D) = \mathcal{D}(\coprod_{x \in X} d, D) \cong \prod_{x \in X} \mathcal{D}(d, D) \cong \prod_{x \in X} UD \cong \mathbf{Set}(X, UD)$$

for any $X \in \mathbf{Set}$ and $D \in \mathcal{D}$. This proves that $F \dashv U$. □

We will now present how Theorem 2.94 can be used to compute coproducts in **Group**. From group theory, we know that any group G admits a generators-and-relations presentation. The generators can be thought of as the "elementary building blocks" of the group, in the sense that every group element can be expressed as a finite word using elements of the generating set and their inverses. However, the collection of formal finite words made up of elements of G and formal inverses is enormous - even if only two generators are used. In any group, there are relations satisfied by the group elements, so that two syntactically different words are actually identified when "evaluated" in the group. This intuitive idea can be made precise in the following way.

Given a set S and a collection of words $R \subset UFS$, we may form the quotient group

$$\langle S \mid R \rangle := FS/\langle\langle R \rangle\rangle,$$

called the group with *generators* S and *relations* R. Moreover, we say that the pair (S, R) defines a *presentation by generators-and-relations* of a group G if

$$G \cong \langle S \mid R \rangle.$$

One of the reasons why this defines a central concept in group theory is that it describes how *every* group is constructed. Indeed, any group G admits a "canonical" presentation; if $\varepsilon_G : FUG \to G$ denotes the surjective evaluation map (which is the counit of the free-forgetful adjunction as seen in Example 2.90 (i)), the first isomorphism theorem implies that

$$G \cong FUG/\ker \varepsilon_G; \tag{2.3}$$

in particular, there is a canonical presentation $\mathrm{Can}(G) := (UG, U \ker \varepsilon_G)$.

Consequently, in order to understand how to form coproducts in **Group**, it suffices to determine how to form coproducts of groups of the form $\langle S \mid R \rangle$. For simplicity, we will show how to form binary coproducts, since the construction immediately generalizes to arbitrary coproducts.

Proposition 2.97. *Let $F \dashv U$ be the free-forgetful adjunction for groups; let $S \in \mathbf{Set}, R \subset UFS$ and $S' \in \mathbf{Set}, R' \subset UFS'$. Then*

$$\langle S \mid R \rangle * \langle S' \mid R' \rangle \cong \langle S \sqcup S' \mid UF\iota_S(R) \cup UF\iota_{S'}(R') \rangle,$$

where $\iota_S \hookrightarrow S \sqcup S'$ and $\iota_{S'} \hookrightarrow S \sqcup S'$ are the coproduct inclusions.

We first construct a category.

Definition 2.98. There is a category **GenRel** whose

- objects are pairs $(S \in \mathbf{Set}, R \subset UFS)$
- a morphism $f : (S, R) \to (S', R')$ is a function $f : S \to S'$ so that $UFf(R) \subset R'$
- given morphisms $(S, R) \xrightarrow{f} (S', R') \xrightarrow{g} (S'', R'')$, the composite is defined to be

$$(S, R) \xrightarrow{g \circ f} (S'', R'');$$

- the identity at (S, R) is $\mathrm{id}_S : (S, R) \to (S, R)$.

Associativity of composition and unitality of the identity follow from the corresponding properties in **Set**.

Proposition 2.99. *There is a functor* $\mathrm{Pres} : \mathbf{GenRel} \to \mathbf{Group}$. *Moreover, it is a left adjoint.*

Proof. Define the action on objects of $\mathrm{Pres} : \mathbf{GenRel} \to \mathbf{Group}$ by

$$\mathrm{Pres}(S, R) := \langle S \mid R \rangle = FS/\langle\langle R \rangle\rangle.$$

For a function $f : S \to S'$, the universal property of the quotient group implies the existence of a unique group homomorphism $\mathrm{Pres}(f) := \overline{Ff} : FS/\langle\langle R \rangle\rangle \to FS'/\langle\langle R' \rangle\rangle$ so that the square

$$\begin{array}{ccc} FS & \xrightarrow{Ff} & FS' \\ {\scriptstyle \pi_{S,R}}\downarrow & & \downarrow{\scriptstyle \pi_{S',R'}} \\ FS/\langle\langle R \rangle\rangle & \xrightarrow[\overline{Ff}]{} & FS'/\langle\langle R' \rangle\rangle \end{array} \qquad \text{commutes .}$$

Functoriality of Pres follows from the functoriality of F.

We now construct a right adjoint Can : **Group** $\to$ **GenRel**. Can carries a group G to its canonical presentation

$$\mathrm{Can}(G) := (UG, U\ker\varepsilon_G)$$

and a group homomorphism $\phi : G \to G'$ to

$$\mathrm{Can}(\phi) := U\phi : U\ker\varepsilon_G \to U\ker\varepsilon_{G'};$$

$U\phi$ indeed defines a moprhism in **GenRel** because the square

$$\begin{array}{ccc} FUG & \xrightarrow{\varepsilon_G} & G \\ {\scriptstyle FU\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ FUG' & \xrightarrow[\varepsilon_{G'}]{} & G' \end{array}$$

commutes by naturality of ε. Functoriality of Can is a consequence of the functoriality of U.

For $(S, R) \in$ **GenRel** and $G \in$ **Group**, we define a bijection

$$\mathbf{Group}(FS/\langle\langle R\rangle\rangle, G) \cong \mathbf{GenRel}((S, R), (UG, U\ker\varepsilon_G)).$$

A group homomorphism $\overline{\phi}$ is mapped to the composite

$$\alpha(\overline{\phi}) : S \xrightarrow{\eta_S} UFS \xrightarrow{U\pi_{S,R}} U(FS/\langle\langle R\rangle\rangle) \xrightarrow{U\overline{\phi}} UG,$$

which defines a morphism in **GenRel** because the top and bottom composites below are equal by naturality of ε and the adjunction $F \dashv U$ triangle identity:

$$\begin{array}{ccccccccc} FS & \xrightarrow{F\eta_S} & FUFS & \xrightarrow{FU\pi_{S,R}} & FU(FS/\langle\langle R\rangle\rangle) & \xrightarrow{FU\overline{\phi}} & FUG & \xrightarrow{\varepsilon_G} & G \\ \| & & \downarrow{\scriptstyle \varepsilon_{FS}} & & \downarrow{\scriptstyle \varepsilon_{FS/\langle\langle R\rangle\rangle}} & & & & \| \\ FS & = & FS & \xrightarrow{\pi_{S,R}} & FS/\langle\langle R\rangle\rangle & & \xrightarrow{\qquad\overline{\phi}\qquad} & & G \end{array} \tag{2.4}$$

.

In the other direction, a morphism $f : (S, R) \to (UG, U\ker\varepsilon_G) \in$ **GenRel** is mapped to (by the universal property of the quotient group) the unique group homomorphism $\beta(f) := \overline{\phi}$ so that the square

$$\begin{array}{ccc} FS & \xrightarrow{Ff} & FUG \\ {\scriptstyle \pi_{S,R}}\downarrow & & \downarrow{\scriptstyle \varepsilon_G} \\ FS/\langle\langle R\rangle\rangle & \xrightarrow{\exists!\overline{\phi}} & G \end{array} \qquad \text{commutes .}$$

Finally, to see why this defines a bijection, we prove that the maps α and β are inverses of each other. For any group homomorphism $\overline{\phi} : FS/\langle\langle R\rangle\rangle \to G$, the identity

$$\beta(\alpha(\overline{\phi})) = \overline{\phi}$$

follows from diagram (2.4). Given a morphism $f : (S, R) \to (UG, U\ker\varepsilon_G) \in$ **GenRel**, naturality of η and the (other) adjunction $F \dashv U$ triangle identity imply that

$$\alpha(\beta(f)) : S \xrightarrow{\eta_S} UFS \xrightarrow{UFf} UFUG \xrightarrow{U\varepsilon_G} UG = S \xrightarrow{f} UG \xrightarrow{\eta_{UG}} UFUG \xrightarrow{U\varepsilon_G} UG = S \xrightarrow{f} UG.$$

We now prove naturality in both variables. For a morphism $g : (S', R') \to (S, R) \in \mathbf{GenRel}$, we need to show that the square

$$\begin{array}{ccc} \mathbf{Group}(FS/\langle\langle R\rangle\rangle, G) & \xrightarrow{\cong} & \mathbf{GenRel}((S,R),(UG, U\ker\varepsilon_G)) \\ \downarrow{\scriptstyle (\overline{Fg})^*} & & \downarrow{\scriptstyle g^*} \\ \mathbf{Group}(FS'/\langle\langle R'\rangle\rangle, G) & \xrightarrow{\cong} & \mathbf{GenRel}((S',R'),(UG, U\ker\varepsilon_G)) \end{array}$$

commutes; this holds because for any test group homomorphism $\overline{\phi} : FS/\langle\langle R\rangle\rangle \to G$:

$$\begin{aligned} & S' \xrightarrow{g} S \xrightarrow{\eta_S} UFS \xrightarrow{UF\pi_{S,R}} U(FS/\langle\langle R\rangle\rangle) \xrightarrow{U\overline{\phi}} UG \\ = {} & S' \xrightarrow{\eta_{S'}} UFS' \xrightarrow{UFg} UFS \xrightarrow{U\pi_{S,R}} U(FS/\langle\langle R\rangle\rangle) \xrightarrow{U\overline{\phi}} UG \\ = {} & S' \xrightarrow{\eta_{S'}} UFS' \xrightarrow{U\pi_{S',R'}} U\big(FS'/\langle\langle R'\rangle\rangle\big) \xrightarrow{U\overline{Fg}} U(FS/\langle\langle R\rangle\rangle) \xrightarrow{U\overline{\phi}} UG. \end{aligned}$$

Naturality in the other variable is immediate. We have thus shown that Pres $\dashv$ Can. □

We can now combine these results to prove Proposition 2.97.

Proof. Using Definition 2.98, we can re-express the hypothesis of Proposition 2.97 as

$$(S, R), (S', R') \in \mathbf{GenRel}.$$

It is easy to show that

$$(S \sqcup S', UF\iota_S(R) \cup UF\iota_{S'}(R')) \cong (S, R) \sqcup (S', R')$$

is the coproduct of (S, R) and (S', R') in **GenRel**. By Proposition 2.99, the functor Pres : **GenRel** $\to$ **Group** is a left adjoint. By Theorem 2.94, left adjoints preserve coproducts. The conclusion now follows:

$$\begin{aligned} \langle S \mid R\rangle * \langle S' \mid R'\rangle &= \mathrm{Pres}(S, R) * \mathrm{Pres}(S', R') \\ &\cong \mathrm{Pres}\big((S, R) \sqcup (S', R')\big) \\ &\cong \mathrm{Pres}(S \sqcup S', UF\iota_S(R) \cup UF\iota_{S'}(R')) \\ &= \left\langle S \sqcup S' \;\middle|\; UF\iota_S(R) \cup UF\iota_{S'}(R')\right\rangle. \end{aligned}$$

□

3 Monads

We have now arrived at the main section of this report: monads. They are a central concept in category theory which often encode algebraic structure in a purely categorical form.

3.1 Monads from adjunctions

Definition 3.1. A *monad* on a category $\mathcal{C}$ is a triple $\mathbf{T} = (T, \eta, \mu)$, where :

- $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor;
- $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ is the unit natural transformation;
- $\mu : T^2 \Rightarrow T$ is the multiplication natural transformation;

so that the following diagrams, expressing the unit and associativity laws respectively, commute:

$$\begin{array}{ccccc} T & \xRightarrow{T\eta} & T^2 & \xLeftarrow{\eta T} & T \\ & \searrow & \Big\Downarrow \mu & \swarrow & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xRightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mu} \\ T^2 & \xRightarrow[\mu]{} & T \end{array} .$$

The following lemma describes how monads naturally arise from adjunctions.

Lemma 3.2. *Any adjunction*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$$

gives rise to a monad $\mathbf{T} = (T, \eta, \mu)$ *on* $\mathcal{C}$*, often referred to as the underyling monad of the adjunction, with*

- $T := UF : \mathcal{C} \to \mathcal{C}$*,*
- *the unit* $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ *of the adjunction serving as the unit of the monad, and*
- *a whiskered version of the counit of the adjunction* $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$*, defined by* $\mu := U\varepsilon F : UFUF \Rightarrow UF$*, serving as the multiplication of the monad.*

Remark 3.3. The underlying monad of an adjunction can be thought of as the "shadow cas by an adjunction", where the data of the adjunction $F \dashv U$ is viewed through the lenses of the category $\mathcal{C}$.

Proof. The triangle identities of the adjunction $F \dashv U$ imply that

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FUF \\ & \searrow & \Big\Downarrow{\scriptstyle \varepsilon F} \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \xRightarrow{\eta U} & UFU \\ & \searrow & \Big\Downarrow{\scriptstyle U\varepsilon} \\ & & U \end{array} \quad \text{commute.}$$

Whiskering the left triangle by U on the left, and the right triangle by F on the right, we obtain the unit law of the monad:

$$\begin{array}{ccccc} UF & \xRightarrow{UF\eta} & UFUF & \xLeftarrow{\eta UF} & UF \\ & \searrow & \Big\Downarrow{\scriptstyle U\varepsilon F} & \swarrow & \\ & & UF & & \end{array} .$$

Moreover, naturality of ε implies that the following square commutes:

$$\begin{array}{ccc} FUFU & \xRightarrow{FU\varepsilon} & FU \\ {\scriptstyle \varepsilon FU}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \varepsilon} \\ FU & \xRightarrow[\varepsilon]{} & \mathrm{id}_{\mathcal{D}} \end{array} .$$

The multiplication law of the monad now follows after whiskering by U on the left and F on the right:

$$\begin{array}{ccc} UFUFUF & \xRightarrow{UFU\varepsilon F} & UFUF \\ {\scriptstyle U\varepsilon FUF}\Big\Downarrow & & \Big\Downarrow{\scriptstyle U\varepsilon F} \\ UFUF & \xRightarrow[U\varepsilon F]{} & UF \end{array} .$$

□

We now give examples of monads; we will mostly apply Lemma 3.2 to examples of adjunctions seen in Example 2.88 and Example 2.90.

Example 3.4. (i) The free monoid adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathbf{Monoid}$$

from Example 2.90 (ii) gives rise to a monad on **Set**. The endofunctor $T = UF : \mathbf{Set} \to \mathbf{Set}$ carries a set S to the underlying set of the free monoid on that set $\bar{S}$, consisting of all finite sequences valued in S. The unit of the monad is the unit of the adjunction: at a set S, its component $\eta_S : S \to TS$ sends an element $s \in S$ to the length-one sequence $s \in TS$. The multiplication of the monad at S is defined by $\mu_S := U\varepsilon_{FS} : T^2S \to TS$ which concatenates a finite sequence of finite sequences $((\mathrm{s}_{11} \cdots \mathrm{s}_{1n_1}) \cdots (\mathrm{s}_{m1} \cdots \mathrm{s}_{mn_m})) \in T^2S$ to obtain a single finite sequence $(\mathrm{s}_{11} \cdots \mathrm{s}_{1n_1} \cdots \mathrm{s}_{m1} \cdots \mathrm{s}_{mn_m}) \in TS$ valued in S. The unit law asserts that, given a finite sequence $(\mathrm{s}_1 \cdots \mathrm{s}_n) \in TS$, the following two processes output that same sequence:

(1) the left triangle applies η to each letter to obtain $((\mathrm{s}_1) \cdots (\mathrm{s}_n)) \in T^2S$, then concatenates to get $(\mathrm{s}_1 \cdots \mathrm{s}_n) \in TS$;
(2) the right triangle applies η to the sequence viewed as a single element of TS to obtain $((\mathrm{s}_1 \cdots \mathrm{s}_n)) \in T^2S$, then concatenates this single word to get $(\mathrm{s}_1 \cdots \mathrm{s}_n) \in TS$.

The associativity law asserts that for a finite sequence of finite sequences of finite sequences

$$(((\mathrm{s}_{111} \cdots \mathrm{s}_{11n_1}) \cdots (\mathrm{s}_{1m_11} \cdots \mathrm{s}_{1m_1n_{m_1}})) \cdots ((\mathrm{s}_{k11} \cdots \mathrm{s}_{11n_k}) \cdots (\mathrm{s}_{km_k1} \cdots \mathrm{s}_{km_kn_{m_k}}))) \in T^3S,$$

the following two resulting elements of TS are the same:

(1) the top-right path concatenates the outer structure first to obtain

$$((\mathrm{s}_{111} \cdots \mathrm{s}_{11n_1}) \cdots (\mathrm{s}_{1m_11} \cdots \mathrm{s}_{1m_1n_{m_1}}) \cdots (\mathrm{s}_{k11} \cdots \mathrm{s}_{11n_k}) \cdots (\mathrm{s}_{km_k1} \cdots \mathrm{s}_{km_kn_{m_k}})) \in T^2S,$$

then concatenates what is left to get

$$(\mathrm{s}_{111} \cdots \mathrm{s}_{11n_1} \cdots \mathrm{s}_{1m_11} \cdots \mathrm{s}_{1m_1n_{m_1}} \cdots \mathrm{s}_{k11} \cdots \mathrm{s}_{11n_k} \cdots \mathrm{s}_{km_k1} \cdots \mathrm{s}_{km_kn_{m_k}}) \in TS;$$

(2) the bottom-left path concatenates the inner structures to obtain

$$((\mathrm{s}_{111} \cdots \mathrm{s}_{11n_1} \cdots \mathrm{s}_{1m_11} \cdots \mathrm{s}_{1m_1n_{m_1}}) \cdots (\mathrm{s}_{k11} \cdots \mathrm{s}_{11n_k} \cdots \mathrm{s}_{km_k1} \cdots \mathrm{s}_{km_kn_{m_k}})) \in T^2S,$$

and then concatenates further to obtain

$$(\mathrm{s}_{111} \cdots \mathrm{s}_{11n_1} \cdots \mathrm{s}_{1m_11} \cdots \mathrm{s}_{1m_1n_{m_1}} \cdots \mathrm{s}_{k11} \cdots \mathrm{s}_{11n_k} \cdots \mathrm{s}_{km_k1} \cdots \mathrm{s}_{km_kn_{m_k}}) \in TS.$$

(ii) The free group adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathbf{Group}$$

from Example 2.88 (i) gives rise to a monad on **Set**. The endofunctor $T = UF : \mathbf{Set} \to \mathbf{Set}$ carries a set S to the underlying set of the free group on S; the unit of the monad is the unit of the adjunction $\eta_S : S \to TS$, and the multiplication of the monad $\mu_S := U\varepsilon_{FS} : T^2S \to TS$ takes a finite word of finite words on S, and applies the necessary inversions to subwords before concatenating to obtain a new finite word constructed using elements of S and their formal inverses. The monad unit and associativity laws hold just as seen in Example 3.4 (i) for **Monoid**, with the only difference that one needs to potentially apply inversion to some subbrackets in the process. If one doesn't want to check the laws manually, that is totally fine - we already know they must hold by Lemma 3.2 above!

(iii) The free R-module adjunction

$$\mathbf{Set} \underset{U}{\overset{R[-]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} {}_R\mathbf{Mod}$$

defines a monad on **Set**. The endofunctor $T := UR[-] : \mathbf{Set} \to \mathbf{Set}$ carries a set S to the set underlying the free R-module $R[S]$, which consists of all formal finite R-linear combinations of elements of S; the unit of the monad $\eta_S : S \to TS$ views an element $s \in S$ as the formal R-linear combination $1 \cdot s \in TS$; the multiplication $\mu_S := U\varepsilon_{R[S]} : T^2S \to TS$ takes an R-linear combination of R-linear combinations of elements of S and distributes the scalars, groups like terms in front of a common $s \in S$ to obtain a new R-linear combination.

(iv) The free pointed set adjunction

$$\mathbf{Set} \underset{U}{\overset{-\sqcup\{-\}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Set}_*$$

defines a monad on **Set**. The endofunctor $(-)_+ := U(- \sqcup \{-\})$ carries a set X to the set $X_+ := X \sqcup \{X\}$; the unit $\eta_X : X \to X_+$ is the inclusion map; the multiplication $\mu_X := U\varepsilon_{(X\sqcup\{X\},X)} : (X_+)_+ \to X_+$ is the identity on X, and maps the two freely-adjoined points on the left to the one on the right. The monad axioms assert the the following diagrams commute:

$$\begin{array}{ccccc} X_+ & \xrightarrow{(\eta_X)_+} & (X_+)_+ & \xleftarrow{\eta_{X_+}} & X_+ \\ & \searrow & \downarrow{\scriptstyle \mu_X} & \swarrow & \\ & & X_+ & & \end{array} \qquad \begin{array}{ccc} ((X_+)_+)_+ & \xrightarrow{\mu_{X_+}} & (X_+)_+ \\ {\scriptstyle (\mu_X)_+}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ (X_+)_+ & \xrightarrow[\mu_X]{} & X_+ \end{array} .$$

Note that when restricted to X, all composites are identities; so it remains to examine what happens to the various basepoints. Labelling the basepoints according to their order of construction by 1, 2 and up to 3 if necessary:

(1) the left triangle action is encoded by

$$\begin{array}{ccccc} 1 & \longrightarrow & 1 & \longrightarrow & 1 \\ & & 2 & & \end{array} ,$$

and the right triangle action by

$$\begin{array}{ccccc} 1 & & 1 & & 1 \\ & \searrow & & \nearrow & \\ & & 2 & & \end{array} ;$$

(2) the top-right composite is encoded by

$$\begin{array}{ccccc} 1 & \longrightarrow & 1 & \longrightarrow & 1 \\ & \nearrow & & \nearrow & \\ 2 & & 2 & & \\ & \nearrow & & & \\ 3 & & & & \end{array} ,$$

and the bottom-left composite by

$$\begin{array}{ccccc} 1 & \longrightarrow & 1 & \longrightarrow & 1 \\ & & & \nearrow & \\ 2 & \longrightarrow & 2 & & \\ & \nearrow & & & \\ 3 & & & & \end{array} .$$

There are also examples of monads that a priori don't arise from adjunctions.

(v) There is a monad on **Set** given by $(\mathcal{P}, \eta, \mu)$, where $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is the covariant power set functor; $\eta : \mathrm{id}_{\mathbf{Set}} \to \mathcal{P}$ is the natural transformation from Example 2.31 (ii), whose component at $A \in \mathbf{Set}$ is $\eta_A : a \in A \mapsto \{a\} \in \mathcal{P}A$; $\mu : \mathcal{P}^2 \Rightarrow \mathcal{P}$ is the natural transformation whose component at $A \in \mathbf{Set}$ is $\mu_A : \{A_i \subset A \mid i \in I\} \in \mathcal{P}^2 A \mapsto \cup_{i \in I} A_i \in \mathcal{P}A$.

(vi) For any monoid $(M, \star, e)$, there is a monad $(- \times M, \eta, \mu)$ on **Set**, where the components of the unit and multiplication at $A \in \mathbf{Set}$ are specified by $\eta_A : a \in A \mapsto (a, e) \in A \times M$ and $\mu_A : (a, m, n) \in A \times M \times M \mapsto (a, m \star n) \in A \times M$. The monad unit axiom uses exactly the monoid unit axiom

$$e \star m = m = m \star e, \ \forall m \in M,$$

and the content of the monad associativity axiom is precisely the monoid associativity axiom

$$(m \star n) \star k = m \star (n \star k), \ \forall m, n, k \in M.$$

3.2 Adjunctions from monads

Lemma 3.2 describes how an adjunction naturally induces a monad. It is therefore natural to ask whether the converse holds: given a monad $\mathbf{T}$ on $\mathcal{C}$, is it possible to construct a category $\mathcal{D}$ as well as a pair of adjoint functors

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D}$$

whose underlying monad is $\mathbf{T}$? We will provide two solutions (see Lemma 3.8 and Lemma 3.11) which have a very special property (see Proposition 3.13).

Definition 3.5. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The *Eilenberg-Moore category*, or category of $\mathbf{T}$-*algebras*, is the category $\mathcal{C}^{\mathbf{T}}$ whose

- objects are pairs $(A \in \mathcal{C}, a : TA \to A)$, called $\mathbf{T}$-*algebras*, so that the following triangle and square commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow^{=} & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad \begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \ ;$$

- morphisms $f : (A, a) \to (B, b)$, also called $\mathbf{T}$-*algebra homomorphisms*, are morphisms $f : A \to B$ in $\mathcal{C}$ so that the following square commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array} \ .$$

Note that if $f : (A, a) \to (B, b)$ and $g : (B, b) \to (C, c)$ are $\mathbf{T}$-algebra homomorphisms, then the composite $g \circ f : A \to C$ in $\mathcal{C}$ defines a $\mathbf{T}$-algebra homomorphism $g \circ f : (A, a) \to (C, c)$, which is defined to be the composite of the two morphisms in $\mathcal{C}^{\mathbf{T}}$. Indeed, pasting the two commutative squares gives us a commutative rectangle:

$$\begin{array}{ccccc} TA & \xrightarrow{Tf} & TB & \xrightarrow{Tg} & TC \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array} \ , \qquad \text{i.e.} \qquad \begin{array}{ccc} TA & \xrightarrow{T(g \circ f)} & TC \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c} \\ A & \xrightarrow[g \circ f]{} & C \end{array} \qquad \text{commutes.}$$

Moreover, for any $\mathbf{T}$-algebra $(A \in \mathcal{C}, a : TA \to A)$, the identity on A defines a $\mathbf{T}$-algebra homomorphism $\mathrm{id}_A : (A, a) \to (A, a)$, which is defined to be $\mathrm{id}_{(A,a)}$ in $\mathcal{C}^{\mathbf{T}}$:

$$\begin{array}{ccc} TA & \xrightarrow{T\mathrm{id}_A} & TA \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow[\mathrm{id}_A]{} & A \end{array} .$$

The unit and associativity of composition laws in $\mathcal{C}^{\mathbf{T}}$ then follow from those in $\mathcal{C}$.

Example 3.6. We apply this notion to the free pointed set monad from Example 3.4 (iv). A $(-)_+$-algebra is a pair $(A, a : A_+ \to A)$ so that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A_+ \\ & \searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (A_+)_+ & \xrightarrow{a_+} & A_+ \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ A_+ & \xrightarrow[a]{} & A \end{array} \qquad \text{commute.}$$

The triangle asserts that $a|_A = \mathrm{id}_A$, which implies that $a : A_+ \to A$ specifies an element $a \in A$ (the image of the freely-adjoined point); while the square gives no additional information since both composites are the identity on A, and map the two new points to $a \in A$. Thus an algebra corresponds to a pointed set. An algebra homomorphism $f : (A, a) \to (B, b)$ is a map $f : A \to B$ so that

$$\begin{array}{ccc} A_+ & \xrightarrow{f_+} & B_+ \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes - that is, $f(a) = b$. This suggests that there is an isomorphism

$$\mathbf{Set}^{(-)_+} \cong \mathbf{Set}_*.$$

Definition 3.7. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. For each $A \in \mathcal{C}$, the free $\mathbf{T}$-algebra on A is $(TA, \mu_A : T^2A \to TA) \in \mathcal{C}^{\mathbf{T}}$; indeed, using most of the defining properties of a monad (except one triangle) from Definition 3.1, the diagrams

$$\begin{array}{ccc} TA & \xrightarrow{\eta_{TA}} & T^2A \\ & \searrow & \downarrow{\scriptstyle \mu_A} \\ & & TA \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T^3A & \xrightarrow{T\mu_A} & T^2A \\ {\scriptstyle \mu_{TA}}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ T^2A & \xrightarrow[\mu_A]{} & TA \end{array} \qquad \text{commute.}$$

For every morphism $f : A \to B \in \mathcal{C}$, there is an induced $\mathbf{T}$-algebra homomorphism $Tf : (TA, \mu_A) \to (TB, \mu_B)$ between the free $\mathbf{T}$-algebras, because

$$\begin{array}{ccc} T^2A & \xrightarrow{T^2f} & T^2B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ TA & \xrightarrow[Tf]{} & TB \end{array}$$

commutes by naturality of $\mu : T^2 \Rightarrow T$. Since T is a functor, this defines a functor $F^{\mathbf{T}} : \mathcal{C} \to \mathcal{C}^{\mathbf{T}}$

The lemma below provides one answer to the question we posed.

Lemma 3.8. *Any monad* $\mathbf{T} = (T, \eta, \mu)$ *on* $\mathcal{C}$ *gives rise to an adjunction*

$$\mathcal{C} \underset{U^{\mathbf{T}}}{\overset{F^{\mathbf{T}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} C^{\mathbf{T}}$$

whose underlying monad is precisely $\mathbf{T}$.

Proof. Define the forgetful functor $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ by

$$\begin{array}{ccc} (A,a) & \overset{U^{\mathbf{T}}}{\longmapsto} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (B,b) & \underset{U^{\mathbf{T}}}{\longmapsto} & B \end{array}$$

Note that by definition we have $U^{\mathbf{T}}F^{\mathbf{T}} = T$. Define the unit of the adjunction to be the unit of the monad $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T = U^{\mathbf{T}}F^{\mathbf{T}}$, and the counit $\varepsilon : F^{\mathbf{T}}U^{\mathbf{T}} \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbf{T}}}$ whose component at $(A, a) \in \mathcal{C}^{\mathbf{T}}$ is $\varepsilon_{(A,a)} : TA \xrightarrow{a} A$.

To see that ε is natural, take a $\mathbf{T}$-algebra homomorphism $f : (A, a) \to (B, b)$ and observe that by definition of a morphism in $\mathcal{C}^{\mathbf{T}}$, the naturality square

$$\begin{array}{ccc} TA & \overset{\varepsilon_{(A,a)}=a}{\longrightarrow} & A \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ TB & \underset{\varepsilon_{(B,b)}=b}{\longrightarrow} & B \end{array}$$

commutes. It follows that $U^{\mathbf{T}}\varepsilon F^{\mathbf{T}} = \mu$.

To conclude, it remains to verify that the specified unit and counit indeed define an adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$; the missing piece of data are the triangle identities

$$\begin{array}{ccc} F^{\mathbf{T}} & \overset{F^{\mathbf{T}}\eta}{\Longrightarrow} & F^{\mathbf{T}}U^{\mathbf{T}}F^{\mathbf{T}} \\ & \searrow & \Downarrow{\scriptstyle \varepsilon F^{\mathbf{T}}} \\ & & F^{\mathbf{T}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} U^{\mathbf{T}} & \overset{\eta U^{\mathbf{T}}}{\Longrightarrow} & U^{\mathbf{T}}F^{\mathbf{T}}U^{\mathbf{T}} \\ & \searrow & \Downarrow{\scriptstyle U^{\mathbf{T}}\varepsilon} \\ & & U^{\mathbf{T}} \end{array} .$$

The left triangle, when evaluated at a component $A \in \mathcal{C}$, follows from the triangle identity defining a monad that we have not yet used; whereas the right triangle, when evaluated at $(A, a) \in \mathcal{C}^{\mathbf{T}}$, commutes due to the unitality axiom of a $\mathbf{T}$-algebra:

$$\begin{array}{ccc} TA & \overset{T\mu_A}{\longrightarrow} & T^2A \\ & \searrow & \downarrow{\scriptstyle \mu_A} \\ & & TA \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \overset{\eta_A}{\longrightarrow} & TA \\ & \searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} .$$

□

Proposition 3.9. *Let* $\mathbf{T} = (T, \eta, \mu)$ *be a monad on a category* $\mathcal{C}$. *The forgetful functor* $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ *reflects isomorphisms.*

Proof. Let $f : (A, a) \to (B, b) \in \mathcal{C}^{\mathbf{T}}$ be a $\mathbf{T}$-algebra homomorphism so that $U^{\mathbf{T}}f = f : A \to B \in \mathcal{C}$ is an isomorphism. Then we see that $f^{-1} : B \to A \in \mathcal{C}$ defines a $\mathbf{T}$-algebra homomorphism $f^{-1} : (B, b) \to (A, a) \in \mathcal{C}^{\mathbf{T}}$ by reversing the top and bottom arrows:

$$\begin{array}{ccc} TA & \overset{Tf}{\longrightarrow} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \underset{f}{\longrightarrow} & B \end{array} \qquad \rightsquigarrow \qquad \begin{array}{ccc} TA & \overset{Tf^{-1}}{\longleftarrow} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \underset{f^{-1}}{\longleftarrow} & B \end{array} .$$

□

We now examine a second construction.

Definition 3.10. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The *Kleisli category* $\mathcal{C}_\mathbf{T}$ is defined so that

- its objects are the objects in $\mathcal{C}$;
- a morphism $A \rightsquigarrow B$ in $\mathcal{C}_\mathbf{T}$ is a morphism $A \to TB$ in $\mathcal{C}$;
- the composite $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$ is defined to be the arrow $A \rightsquigarrow C$ in $\mathcal{C}_\mathbf{T}$ whose corresponding arrow in $\mathcal{C}$ is
$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC;$$
- the identity on $A \in \mathcal{C}_\mathbf{T}$ is defined to be $\eta_A : A \to TA$ in $\mathcal{C}$.

The following diagram, which uses naturality of μ as well as the monad associativity law, demonstrates the associativity axiom of the composite of three morphisms $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C \overset{h}{\rightsquigarrow} D$.

$$\begin{array}{ccccccccccc}
A & \xrightarrow{f} & TB & \xrightarrow{Tg} & T^2C & \xrightarrow{\mu_C} & TC & \xrightarrow{Th} & T^2D & \xrightarrow{\mu_D} & TD \\
\| & & & & \| & & & & \| & & \| \\
 & & & & T^2C & \xrightarrow{T^2h} & T^3D & \xrightarrow{\mu_{TD}} & T^2D & \xrightarrow{\mu_D} & TD \\
 & & & & \| & & \| & & & & \| \\
A & \xrightarrow{f} & TB & \xrightarrow{Tg} & T^2C & \xrightarrow{T^2h} & T^3D & \xrightarrow{T\mu_D} & T^2D & \xrightarrow{\mu_D} & TD
\end{array}$$

The identity axiom of a category follows from the naturality of η and its unit property as the unit component of the monad.

$$A \xrightarrow{\eta_A} TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB = A \xrightarrow{f} TB \xrightarrow{\eta_{TB}} T^2B \xrightarrow{\mu_B} TB = A \xrightarrow{f} TB$$

$$A \xrightarrow{f} TB \xrightarrow{T\eta_B} T^2B \xrightarrow{\mu_B} TB = A \xrightarrow{f} TB \xrightarrow{\mathrm{id}_{TB}} TB = A \xrightarrow{f} TB$$

This proves that $\mathcal{C}_\mathbf{T}$ indeed forms a category.

The lemma below provides a second answer to the question we posed.

Lemma 3.11. *For any monad $\mathbf{T} = (T, \eta, \mu)$ acting on a category $\mathcal{C}$, there is an adjunction*

$$\mathcal{C} \underset{U_\mathbf{T}}{\overset{F_\mathbf{T}}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} C_\mathbf{T}$$

whose underyling monad is $\mathbf{T}$.

Proof. Define a functor $F_\mathbf{T} : \mathcal{C} \to \mathcal{C}_\mathbf{T}$ that takes an object $A \in \mathcal{C}$ to $A \in \mathcal{C}_\mathbf{T}$, and carries a morphism $f : A \to B \in \mathcal{C}$ to the morphism $A \overset{F_\mathbf{T} f}{\rightsquigarrow} B \in \mathcal{C}_\mathbf{T}$ defined by either composite (by naturality of η): $A \xrightarrow{f} B \xrightarrow{\eta_B} TB = A \xrightarrow{\eta_A} tA \xrightarrow{Tf} TB$.

To prove functoriality, given maps $A \xrightarrow{f} B \xrightarrow{g} C$, observe that naturality of η and the monad unit law imply

$$\begin{aligned}
F_\mathbf{T}(g) \circ F_\mathbf{T}(f) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} T^2C \xrightarrow{\mu_C} TC \\
&= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \xrightarrow{\mathrm{id}_{TC}} TC \\
&= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \\
&= A \xrightarrow{g \circ f} C \xrightarrow{\eta_C} TC \\
&= F_\mathbf{T}(g \circ f),
\end{aligned}$$

and the definition of the identity in $\mathcal{C}_{\mathbf{T}}$ gives

$$\begin{aligned} F_{\mathbf{T}}(\mathrm{id}_A \in \mathcal{C}) &= A \xrightarrow{\mathrm{id}_A} A \xrightarrow{\eta_A} TA \\ &= A \xrightarrow{\eta_A} TA \\ &= \mathrm{id}_{F_{\mathbf{T}}A} \in \mathcal{C}_{\mathbf{T}}. \end{aligned}$$

We also have a functor $U_{\mathbf{T}} : \mathcal{C}_{\mathbf{T}} \to \mathcal{C}$ which takes an object $A \in \mathcal{C}_{\mathbf{T}}$ to $TA \in \mathcal{C}$, and carries a morphism $A \overset{f}{\rightsquigarrow} B$ to the composite $TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB$.

This assignment is functorial because given two morphisms $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$:

$$\begin{aligned} U_{\mathbf{T}}(g \circ f) &= TA \xrightarrow{Tf} T^2B \xrightarrow{T^2g} T^3C \xrightarrow{T\mu_C} T^2C \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} T^2B \xrightarrow{T^2g} T^3C \xrightarrow{\mu_{TC}} T^2C \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC \\ &= U_{\mathbf{T}}(g) \circ U_{\mathbf{T}}(f). \end{aligned}$$

where we used the monad associativity law as well as naturality of μ. Given an object $A \in \mathcal{C}_{\mathbf{T}}$, identity preservation comes from the monad unit law:

$$TA \xrightarrow{T\eta_C} T^2A \xrightarrow{\mu_A} TA = \mathrm{id}_{TA}.$$

With these definitions, we see that $U_{\mathbf{T}}F_{\mathbf{T}} = T$. Our next task is to show that $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$. For $A \in \mathcal{C}$ and $B \in \mathcal{C}_{\mathbf{T}}$, the definitions of morphisms gives an evident "identity" bijection

$$\mathcal{C}_{\mathbf{T}}(F_{\mathbf{T}}A, B) \xrightarrow{\cong} \mathcal{C}(A, U_{\mathbf{T}}B),$$

To prove naturality, consider morphisms $A' \xrightarrow{h} A \in \mathcal{C}$, $B \overset{k}{\rightsquigarrow} B' \in \mathcal{C}_{\mathbf{T}}$. The naturality square

$$\begin{array}{ccc} \mathcal{C}_{\mathbf{T}}(F_{\mathbf{T}}A, B) & \xrightarrow{\cong} & \mathcal{C}(A, U_{\mathbf{T}}B) \\ \downarrow{\scriptstyle (F_{\mathbf{T}}h)^*} & & \downarrow{\scriptstyle h^*} \\ \mathcal{C}_{\mathbf{T}}(F_{\mathbf{T}}A', B) & & \mathcal{C}(A', U_{\mathbf{T}}B) \\ \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle (U_{\mathbf{T}}k)_*} \\ \mathcal{C}_{\mathbf{T}}(F_{\mathbf{T}}A', B') & \xrightarrow[\cong]{} & \mathcal{C}(A', U_{\mathbf{T}}B') \end{array}$$

commutes because for any test morphism $A \overset{f}{\rightsquigarrow} B$ in the top left hom-set, the following diagram is commutative:

$$\begin{array}{ccccccccccc} A' & \xrightarrow{h} & A & \xrightarrow{\eta_A} & TA & \xrightarrow{Tf} & T^2B & \xrightarrow{\mu_B} & TB & \xrightarrow{Tk} & T^2B' & \xrightarrow{\mu_{B'}} & TB' \\ \| & & \| & & & & \| & & \| & & \| & & \| \\ A' & \xrightarrow{h} & A & \xrightarrow{f} & TB & \xrightarrow{\eta_{TB}} & T^2B & \xrightarrow{\mu_B} & TB & \xrightarrow{Tk} & T^2B' & \xrightarrow{\mu_{B'}} & TB' \\ \| & & \| & & \| & & & & \| & & \| & & \| \\ A' & \xrightarrow{h} & A & \xrightarrow{f} & TB & & \xrightarrow{\mathrm{id}_{TB}} & & TB & \xrightarrow{Tk} & T^2B' & \xrightarrow{\mu_{B'}} & TB' \end{array}.$$

Having proven the adjunction, we can now recover the unit η' and counit ε' associated to $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$. We see that for $A \in \mathcal{C}$, η'_A is the transpose of $\mathrm{id}_{F_{\mathbf{T}}A} \in \mathcal{C}_{\mathbf{T}}$, which was defined to be

precisely η_A; likewise, for $B \in \mathcal{C}_{\mathbf{T}}$, $\varepsilon'_B : TB \rightsquigarrow B$ is the transpose of $\mathrm{id}_{TB} \in \mathcal{C}$. Thus, if we denote by μ' the multiplication of the underlying monad, for $A \in \mathcal{C}$ we compute

$$\mu'_A = U_{\mathbf{T}}\varepsilon'_{F_{\mathbf{T}}A} = T^2A \xrightarrow{T\mathrm{id}_{TA}} T^2A \xrightarrow{\mu_A} TA = \mu_A$$

We have thus proven that the underlying monad is $\mathbf{T} = (T, \eta, \mu)$, as desired. □

Definition 3.12. For a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{C}$, there is a category $\mathrm{Adj}_{\mathbf{T}}$ defined in the following way:

- an object is a fully specified adjunction

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D} \qquad \eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF,\ \varepsilon : UF \Rightarrow \mathrm{id}_{\mathcal{D}}$$

  so that its underlying monad is $\mathbf{T}$;

- a morphism between two adjunctions is a functor $K : \mathcal{D} \to \mathcal{D}'$ commuting with the left and right adjunctions:

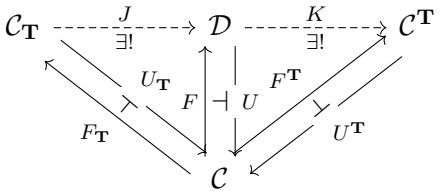
$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ {\scriptstyle F}\uparrow \dashv \downarrow{\scriptstyle U} & & {\scriptstyle F'}\uparrow \dashv \downarrow{\scriptstyle U'} \\ \mathcal{C} & = & \mathcal{C} \end{array}$$

  We will often display a morphism as a triangle rather than a square with one side being the identity, but this presentation allows us to see that K is a strict morphism of adjunctions as defined in Proposition 2.91.

- Composition is the composition of functors, and the identity is the identity functor.

The proposition below explains in what sense the Eilenberg-Moore and Kleisli adjunctions stand out among other potential adjunctions giving rise to a given monad.

Proposition 3.13. *Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The Kleisli adjunction $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$ is initial in $\mathrm{Adj}_{\mathbf{T}}$ and the Eilenberg-Moore adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$ is terminal. That is, for any adjunction $F \dashv U$ inducing the monad $\mathbf{T}$, there exist unique functors $J : \mathcal{C}_{\mathbf{T}} \to D$ and $K : D \to \mathcal{C}^{\mathbf{T}}$ commuting with the left and right adjunctions:*



Proof. Throughout the proof, we denote by ε the counit of the adjunction $F \dashv U$.

The Kleisli adjunction is initial. Assume that a functor $J : \mathcal{C}_{\mathbf{T}} \to \mathcal{D}$ commuting with the left and right adjoints exists. The condition that $JF^{\mathbf{T}} = F$ forces us to define $JX = FX$ for $X \in \mathcal{C}_{\mathbf{T}}$. By Proposition 2.91, if J is a strict morphism of adjunctions, then the following square commutes for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}_{\mathbf{T}}$:

$$\begin{array}{ccc} \mathcal{C}_{\mathbf{T}}(F_{\mathbf{T}}X, Y) & \xrightarrow{J} & \mathcal{D}(JF_{\mathbf{T}}X, JY) = \mathcal{D}(FX, JY = FY) \\ \cong\downarrow{\scriptstyle F_{\mathbf{T}} \dashv U_{\mathbf{T}}} & & \cong\downarrow{\scriptstyle F \dashv U} \\ \mathcal{C}(X, U_{\mathbf{T}}Y) & = & \mathcal{C}(X, UJY). \end{array} \tag{3.1}$$

This shows that J must carry a morphism $f : X \rightsquigarrow Y$ in $\mathcal{C}^{\mathbf{T}}$ to the composite

$$Jf = FX \xrightarrow{Ff} FUFY \xrightarrow{\varepsilon_{FY}} FY.$$

Note that if J is defined in this way, then we do indeed have $JF^{\mathbf{T}} = F$ and $UJ = U^{\mathbf{T}}$. Both identities are obvious on objects.

For the first equality, a morphism $f : X \to Y \in \mathcal{C}$ is carried via $F^{\mathbf{T}}$ to $X \rightsquigarrow Y \in \mathcal{C}^{\mathbf{T}}$ defined by

$$X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY = UFY,$$

which is then mapped via J to the composite

$$FX \xrightarrow{Ff} FY \xrightarrow{F\eta_Y} FUFY \xrightarrow{\varepsilon_{FY}} FY,$$

which is equal to Ff due to one of the triangle identities defining $F \dashv U$.

For the second equality, a morphism $g : X \rightsquigarrow Y \in \mathcal{C}^{\mathbf{T}}$ is carried via J to the composite

$$FX \xrightarrow{Fg} FUFY \xrightarrow{\varepsilon_{FY}} FY,$$

and then via U to the composite

$$UFX \xrightarrow{UFg} UFUFY \xrightarrow{U\varepsilon_{FY}} UFY.$$

Using the identities $T = UF$ and $\mu = U\varepsilon F$, the commutative diagram below

$$\begin{array}{ccccc} UFX & \xrightarrow{UFg} & UFUFY & \xrightarrow{U\varepsilon_{FY}} & UFY \\ \| & & \| & & \| \\ TX & \xrightarrow[Tg]{} & T^2Y & \xrightarrow[\mu_Y]{} & TY \end{array}$$

identifies the top composite with the bottom composite

$$TX \xrightarrow{Tg} T^2Y \xrightarrow{\mu_Y} TY,$$

which by definition is equal to $U^{\mathbf{T}}g$.

Finally, we need to prove that this definition of J on objects and morphisms in $\mathcal{C}^{\mathbf{T}}$ is functorial. For morphisms $X \overset{f}{\rightsquigarrow} Y \overset{g}{\rightsquigarrow} Z$, the following diagram, which uses the naturality of ε, whose top composite is $J(g \circ f)$ and bottom composite $Jg \circ Jf$, shows that they are equal:

$$\begin{array}{ccccccccc} FX & \xrightarrow{Ff} & FUFY & \xrightarrow{FUFg} & FUFUFZ & \xrightarrow{FU\varepsilon_{FZ}} & FUFZ & \xrightarrow{\varepsilon_{FZ}} & FZ \\ \| & & \| & & & \searrow^{\varepsilon_{FUFZ}} & & & \| \\ FX & \xrightarrow[Ff]{} & FUFY & \xrightarrow[\varepsilon_{FY}]{} & FY & \xrightarrow[Fg]{} & FUFZ & \xrightarrow[\varepsilon_{FZ}]{} & FZ \end{array}.$$

Moreover, the identity on $X \in \mathcal{C}^{\mathbf{T}}$ is mapped to $FX \xrightarrow{F\eta_X} FUFX \xrightarrow{\varepsilon_{FX}} FX$, which, by one of the two triangle identities for $F \dashv U$, is id_{FX}.

The Eilenberg-Moore adjunction is terminal. Assume that there is a functor $K : \mathcal{D} \to \mathcal{C}^{\mathbf{T}}$ commuting with the left and right adjoints. The condition $U^{\mathbf{T}}K = U$ forces the action of K on objects and morphisms to be that of U. It remains to investigate what the structure map $UFUD \to UD$ of KD should be, for $D \in \mathcal{D}$. We know that if we are given a $\mathbf{T}$-algebra $(C, c) \in \mathcal{C}^{\mathbf{T}}$, we can recover the strucure map using the adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$; indeed, the counit

of this adjunction at (C, c) was defined precisely to be c. With this in mind, and using the fact that K is a strict morphism of adjunctions, the diagram

$$\begin{array}{ccc} \mathcal{D}(FC, D) & \xrightarrow{K} & \mathcal{C}^{\mathbf{T}}(KFC, KD) = \mathcal{C}^{\mathbf{T}}(F^{\mathbf{T}}C, KD) \\ \cong\downarrow{\scriptstyle F\dashv U} & & \cong\downarrow{\scriptstyle F^{\mathbf{T}}\dashv U^{\mathbf{T}}} \\ \mathcal{C}(C, UD) & = & \mathcal{C}(C, U^{\mathbf{T}}KD) \end{array}$$

commutes and thus we obtain $KD = (UD, U\varepsilon_D : UFUD \to UD)$.

Conversely, if we define K in this manner, we need to verify if it is well-defined and functorial. For $D \in \mathcal{D}$, we check if $(UD, U\varepsilon_D)$ is a $\mathbf{T}$-algebra. Note that naturality of ε applied to its own component $\varepsilon_D : FUD \to D$ yields a commuting diagram

$$\begin{array}{ccc} FUFUD & \xrightarrow{FU\varepsilon_D} & FUD \\ {\scriptstyle \varepsilon_{FUD}}\downarrow & & \downarrow{\scriptstyle \varepsilon_D} \\ FUD & \xrightarrow[\varepsilon_D]{} & D \end{array} .$$

Together with one triangle identity of $F \dashv U$, as well as applying U to the square above, we conclude:

$$\begin{array}{ccc} UD & \xrightarrow{\eta_{UD}} & UFUD \\ & \searrow & \downarrow{\scriptstyle U\varepsilon_D} \\ & & UD \end{array} \quad \text{and} \quad \begin{array}{ccc} UFUFUDA & \xrightarrow{UFU\varepsilon_D} & UFUD \\ {\scriptstyle \mu_{UD}=U\varepsilon_{FUD}}\downarrow & & \downarrow{\scriptstyle U\varepsilon_D} \\ UFUD & \xrightarrow[U\varepsilon_D]{} & UD \end{array} .$$

In addition, a morphism $g : D \to D' \in \mathcal{D}$ is indeed mapped to a $\mathbf{T}$-algebra homomorphism $Ug : (UD, U\varepsilon_D) \to (UD', U\varepsilon'_D)$ by naturality of ε:

$$\begin{array}{ccc} UFUD & \xrightarrow{UFUg} & UFUD' \\ {\scriptstyle U\varepsilon_D}\downarrow & & \downarrow{\scriptstyle U\varepsilon'_D} \\ UD & \xrightarrow[Ug]{} & UD' \end{array} .$$

Having verified all this, functoriality of K follows automatically from that of U. Finally, the identities $KF = F^{\mathbf{T}}$ and $U^{\mathbf{T}}K = U$ follow directly from the definitions. This concludes the proof. □

Example 3.14. Consider the free group monad from Example 3.4 (ii). We show that the canonical comparison functor $K : \mathbf{Group} \to \mathbf{Set}^{\mathbf{T}}$ from Proposition 3.13 is an isomorphism. Recall that K carries a group G to $(UG, U\varepsilon_G) \in \mathbf{Set}^{\mathbf{T}}$, and a group homomorphism $\phi : G \to G'$ to $U\phi : (UG, U\varepsilon_G) \to (UG', U\varepsilon'_G)$. For the other direction, suppose we are given a $\mathbf{T}$-algebra $(X, \xi : TX \to X)$, from which we aim to construct a group. By definition of a $\mathbf{T}$-algebra, we have commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \searrow & \downarrow{\scriptstyle \xi} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} T^2X & \xrightarrow{T\xi} & TX \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \xi} \\ TX & \xrightarrow[\xi]{} & X \end{array} .$$

Using the map ξ, we will define all the components of a group. Define $* : X \times X \to X$ by $x * y := \xi(xy)$ (where $xy \in TX$ is the two-letter word), $e := \xi(1 \in TX)$ and $x^{-1} := \xi(x^{-1} \in TX)$. We will now use the two diagrams to demonstrate that the group axioms hold. Note that

the commutative triangle tells us that $\xi(x \in TX) = x$. To show associativity of $*$, take three elements $x, y, z \in X$ and observe that

$$(x*y)*z = \xi\circ T\xi((xy)(z)) = \xi\circ\mu_X((xy)(z)) = \xi(xyz) = \xi\circ\mu_X((x)(yz)) = \xi\circ T\xi((x)(yz)) = x*(y*z).$$

For unitality of e:

$$e * x = \xi \circ T\xi((1)(x)) = \xi \circ \mu_X((1)(x)) = x = \xi \circ \mu_X((x)(1)) = \xi \circ T\xi((x)(1)) = x * e$$

Finally, the inverse axiom holds:

$$x^{-1}*x = \xi\circ T\xi((x^{-1})(x)) = \xi\circ\mu_X((x^{-1})(x)) = e = \xi\circ\mu_X((x)(x^{-1})) = \xi\circ T\xi((x)(x^{-1})) = x*x^{-1},$$

as desired. We thus obtain a group $(X, *, e, (-)^{-1})$. Now, an algebra homomorphism $f : (X, \xi) \to (X', \xi')$ satisfies

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle x i'} \\ X & \xrightarrow[f]{} & X' \end{array} .$$

Evaluating both composites at the two-letter word $x, y \in TX$, we discover that

$$f(x * y) = f(x) * f(y).$$

Plugging in $1 \in TX$ and $x^{-1} \in TX$ respectively, we obtain

$$f(e) = e'$$

and

$$f(x^{-1}) = f(x)^{-1}.$$

This shows that f induces a group homomorphism between the groups corresponding to the $\mathbf{T}$-algebras. A direct computation shows that the constructed assignment $\mathbf{Set}^{\mathbf{T}} \to \mathbf{Group}$ on objects and morphisms provides an inverse to $K : \mathbf{Group} \to \mathbf{Set}^{\mathbf{T}}$ (note that we do not need to verify functoriality!), and thus K is an isomorphism of categories.

Remark 3.15. The same reasoning as in Example 3.14 can be used to prove the same result for the free monoid monad from Example 3.4 (i) as well as the free R-module monad from Example 3.4 (iii) (and hence also for the free vector space and free abelian group monads).

By Proposition 3.13, there is in particular a unique morphism of adjunctions $K : \mathcal{C}_{\mathbf{T}} \to \mathcal{C}^{\mathbf{T}}$ between the Kleisli adjunction $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$ and the Eilenberg-Moore adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$. The following lemma explores some of its properties.

Lemma 3.16. *Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The canonical functor $K : \mathcal{C}_{\mathbf{T}} \to \mathcal{C}^{\mathbf{T}}$ is fully faithful. Moreover, its image on objects is the collection of free $\mathbf{T}$-algebras.*

Proof. Proposition 3.13 explicitly describes K (in fact, in two ways, which turn out to be, and have to be, equivalent). K carries an object $X \in \mathcal{C}_{\mathbf{T}}$ to the free $\mathbf{T}$-algebra (TX, μ_X) and a morphism $f : X \rightsquigarrow Y \in \mathcal{C}_{\mathbf{T}}$ to the composite

$$(TX, \mu_X) \xrightarrow{Tf} (T^2Y, \mu_{TY}) \xrightarrow{\mu_Y} (TY, \mu_Y).$$

Fully faithfulness follows from a more general remark. The proof of Proposition 3.13 displays the commutative diagram (3.1) associated with the strict morphism of adjunctions J; since any $X \in \mathcal{C}_{\mathbf{T}}$ can be viewed as an object $X \in \mathcal{C}$ with the property that $F_{\mathbf{T}}X = X$, the displayed isomorphisms prove that J is always fully faithful. The same observation does not hold for K in general. However, in our situation, $J = K$. Thus we conclude that K is fully faithful. □

Remark 3.17. Lemma 3.16 tells us in particular when K defines an equivalence between the Kleisli and Eilenberg-Moore categories: whenever K is essentially surjective on objects (by Theorem 2.43), or in other words, when every $\mathbf{T}$-algebra is isomorphic to a free $\mathbf{T}$-algebra.

Example 3.18. Consider the free pointed set monad from Example 3.4 (iv). Every $(-)_+$-algebra (A, a) is isomorphic to a free $(-)_+$-algebra $(B_+, \mu_B : (B_+)_+ \to B_+)$, where $B := A \setminus \{a\}$, via the $(-)_+$-algebra isomorphism $f : A \to B_+$ which is the identity on B and maps $a \in A$ to the freely-adjointed point in B_+. In particular, the Kleisli and Eilenberg-Moore categories are equivalent to $\mathbf{Set}_*$.

3.3 Uniqueness of categories of algebras

The goal of this subsection, due to the present author, is to prove Corollary 3.27: if a functor admits two left adjoints, then the Eilenberg-Moore categories on the monads induced by each adjunction are (compatibly) isomorphic.

We start by defining a category of monads.

Definition 3.19. There is a category $\mathbf{Monad}^{\mathrm{co}}$ defined as follows:

- an object of this category is a pair $(\mathcal{C}, \mathbf{T})$, where $\mathbf{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$;
- a morphism $(\mathcal{C}, \mathbf{T}) \to (\mathcal{C}', \mathbf{T}')$ is a pair (F, α), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $\alpha : T'F \Rightarrow FT$ is a natural transformation, satisfying the unit and multiplication compatibility axioms:

$$\begin{array}{ccc} F & \xRightarrow{\eta' F} & T'F \\ {\scriptstyle F\eta}\Downarrow & \swarrow_{\alpha} & \\ FT & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} T'^2F & \xRightarrow{T'\alpha} & T'FT & \xRightarrow{\alpha T} & FT^2 \\ {\scriptstyle \mu' F}\Downarrow & & & & \Downarrow{\scriptstyle F\mu} \\ T'F & & \xRightarrow[\alpha]{} & & FT \end{array} ;$$

- the composite of two morphisms $(\mathcal{C}, \mathbf{T}) \xrightarrow{(F,\alpha)} (\mathcal{C}', \mathbf{T}') \xrightarrow{(F',\alpha')} (\mathcal{C}'', \mathbf{T}'')$ is defined to be $(F'F, F'\alpha \circ \alpha' F)$. To see why this is well-defined, note first that $F'F : \mathcal{C} \to \mathcal{C}''$ and

$$F'\alpha \circ \alpha' F : T''F'F \xRightarrow{\alpha' F} F'T'F \xRightarrow{F'\alpha} F'FT.$$

  Moreover, the unit compatibility axioms of the two monads express the relations

$$\alpha \circ \eta' F = F\eta \qquad \text{and} \qquad \alpha' \circ \eta'' F' = F'\eta',$$

  which assemble to give

$$\begin{aligned} F'\alpha \circ \alpha' F \circ \eta'' F'F &= F'\alpha \circ F'\eta' F \\ &= F'F\eta, \end{aligned}$$

  as desired.

  Finally, the multiplication compatibility axioms state that

$$F\mu \circ \alpha T \circ T'\alpha = \alpha \circ \mu' F \qquad \text{and} \qquad F'\mu' \circ \alpha' T' \circ T''\alpha' = \alpha' \circ \mu'' F'.$$

  Together with the naturality of α' in the first step, they assemble into

$$\begin{aligned} F'F\mu \circ F'\alpha T \circ \alpha' FT \circ T''F'\alpha \circ T''\alpha' F &= F'F\mu \circ F'\alpha T \circ F'T'\alpha \circ \alpha' T'F \circ T''\alpha' F \\ &= F'\alpha \circ F'\mu' F \circ \alpha' T'F \circ T''\alpha' F \\ &= F'\alpha \circ \alpha' F \circ \mu'' F'F; \end{aligned}$$

- the identity on $(\mathcal{C}, \mathbf{T})$ is $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_T)$.

  Finally, we need to check associativity of composition and identity axioms. The latter is immediate, so we focus on the former. Given a composable triple of morphisms $(\mathcal{C}, \mathbf{T}) \xrightarrow{(F,\alpha)} (\mathcal{C}', \mathbf{T}') \xrightarrow{(F',\alpha')} (\mathcal{C}'', \mathbf{T}'') \xrightarrow{(F'',\alpha'')} (\mathcal{C}''', \mathbf{T}''')$, a direct computation shows that both composites are equal to

$$(F''F'F, F''F'\alpha \circ F''\alpha'F \circ \alpha''F'F).$$

Remark 3.20. In the first attempt at defining a suitable category of monads (which was called **Monad**) for our goal (that is, to prove Corollary 3.27), the data defining an object involved a natural transformation $\alpha : FT \Rightarrow T'F$, rather than a natural transformation pointing in the opposite direction as in Definition 3.19. This choice does define a category **Monad**, but it does not enable us to define a functor analogous to the one in Proposition 3.24. This is the reason behind the superscript "co" in $\mathbf{Monad}^{co}$.

Next, we define a category of adjunctions, where the domain and codomain categories are fixed.

Definition 3.21. Let $\mathcal{C}, \mathcal{D}$ be categories. There is a category $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$ specified by the following data:

- objects are fully-specified adjunctions

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D} \qquad \eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF,\ \varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}};$$

- a morphism $(F, U, \eta, \varepsilon) \to (F', U', \eta', \varepsilon')$ is a pair of natural transformations (α, β) where $\alpha : F \Rightarrow F'$, $\beta : U \Rightarrow U'$ so that the following unit and counit compatibility conditions hold:

$$\eta' = U'\alpha \circ \beta F \circ \eta \qquad \text{and} \qquad \varepsilon = \varepsilon' \circ F'\beta \circ \alpha U;$$

- the composite of two morphisms $(F, U, \eta, \varepsilon) \xrightarrow{(\alpha,\beta)} (F', U', \eta', \varepsilon') \xrightarrow{(\alpha',\beta')} (F'', U'', \eta'', \varepsilon'')$ is computed by compositing the natural transformations in each coordinate :

$$(F, U, \eta, \varepsilon) \xrightarrow{(\alpha'\circ\alpha, \beta'\circ\beta)} (F'', U'', \eta'', \varepsilon'').$$

  This is well-defined due to the following computations, which use naturality of β' and α' respectively:

$$\begin{aligned} \eta'' &= U''\alpha' \circ \beta' F' \circ \eta' \\ &= U''\alpha' \circ U\alpha \circ \beta' F \circ \beta F \circ \eta \\ &= U''(\alpha' \circ \alpha) \circ (\beta' \circ \beta)F \circ \eta, \end{aligned}$$

  and

$$\begin{aligned} \varepsilon &= \varepsilon' \circ F'\beta \circ \alpha U \\ &= \varepsilon'' \circ F''\beta' \circ \alpha' U' \circ F'\beta \circ \alpha U \\ &= \varepsilon'' \circ F''\beta' \circ F''\beta \circ \alpha' U \circ \alpha U \\ &= \varepsilon'' \circ F''(\beta' \circ \beta) \circ (\alpha' \circ \alpha)U; \end{aligned}$$

- the identity on $(F, U, \eta, \varepsilon)$ is $(\mathrm{id}_F, \mathrm{id}_U)$, which trivially satisfy the unit and counit compatibility axioms.

  Here the associativity and identity axioms of a category follow from the corresponding properties in the functor categories $\mathcal{D}^{\mathcal{C}}$ and $\mathcal{C}^{\mathcal{D}}$.

The proposition below provides a useful caracterization of isomorphisms in $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$.

Proposition 3.22. *Let $\mathcal{C}, \mathcal{D}$ be categories. Then there is a forgetful functor $\mathcal{U} : \mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}} \to \mathcal{D}^{\mathcal{C}} \times \mathcal{C}^{\mathcal{D}}$ that reflects isomorphisms.*

Proof. Functoriality of $\mathcal{U}$ follows from the definition of compositon and identities in $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$. To show reflection of isomorphisms, consider a morphism $(\alpha, \beta) : (F, U, \eta, \varepsilon) \to (F', U', \eta', \varepsilon') \in \mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$ so that both components are isomorphisms. The defining property of a morphism states that $\eta' = U'\alpha \circ \beta F \circ \eta$ and $\varepsilon = \varepsilon' \circ F'\beta \circ \alpha U$. Rearranging both expressions using inversion as well and exploiting naturality of β^{-1} and α^{-1} respectively, we obtain $\eta = \beta^{-1}F \circ U'\alpha^{-1} = U\alpha^{-1} \circ \beta^{-1}F'$ and $\varepsilon' = \varepsilon \circ \alpha^{-1}U \circ F'\beta^{-1} = \varepsilon \circ F\beta^{-1} \circ \alpha^{-1}U'$. This proves that $(\alpha^{-1}, \beta^{-1})$ defines a morphism in $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$, which provides an inverse to (α, β), as desired. □

Remark 3.23. Suppose a functor $U : \mathcal{D} \to \mathcal{C}$ admits two left adjoints; that is, we have two objects $(F, U, \eta, \varepsilon), (F', U, \eta', \varepsilon') \in \mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$. By Proposition 2.93, the natural transformation $\alpha : F \overset{F\eta'}{\Longrightarrow} FUF' \overset{\varepsilon F'}{\Longrightarrow} F'$ is a natural isomorphism. Moreover, (α, id_U) defines a morphism between these objects; the adjunction triangle identities, naturality of η and ε imply

$$U\varepsilon F' \circ UF\eta' \circ \eta = U\varepsilon F' \circ \eta UF' \circ \eta' = \eta'$$

and

$$\varepsilon' \circ \varepsilon F'U \circ F\eta' U = \varepsilon \circ FU\varepsilon' \circ F\eta' U = \varepsilon.$$

In particular, Proposition 3.22 implies that $(\alpha, \mathrm{id}_U) : (F, U, \eta, \varepsilon) \overset{\cong}{\to} (F', U, \eta', \varepsilon')$ defines an isomorphism in $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$.

The following proposition provides an arrow from the (opposite) category of adjunctions to the category of monads.

Proposition 3.24. *For any pair of categories $(\mathcal{C}, \mathcal{D})$, there is a functor*

$$\mathcal{M}^{\mathrm{co}} : (\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}})^{\mathrm{op}} \to \mathbf{Monad}^{\mathrm{co}}.$$

Proof. $\mathcal{M}^{\mathrm{co}}$ carries an adjunction $(F, U, \eta, \varepsilon)$ to its underlying monad $(\mathcal{C}, UF, \eta, U\varepsilon F)$, and a morphism

$$(\alpha, \beta) : (F, U, \eta, \varepsilon) \to (F', U', \eta', \varepsilon')$$

in $(\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}})^{\mathrm{op}}$, equivalently a morphism

$$(\alpha, \beta) : (F', U', \eta', \varepsilon') \to (F, U, \eta, \varepsilon)$$

in $\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}}$, to the pair

$$(\mathrm{id}_{\mathcal{C}}, \gamma = U\alpha \circ \beta F').$$

We first show it is well-defined; that is, we need to prove that $(\mathrm{id}_{\mathcal{C}}, \gamma)$ does indeed define a morphism of monads

$$(\mathcal{C}, UF, \eta, U\varepsilon F) \to (\mathcal{C}, U'F', \eta', U'\varepsilon' F').$$

The unit compatibility condition holds due to the unit compatibility condition of (α, β):

$$U\alpha \circ \beta F' \circ \eta' = \eta.$$

To prove the multiplication compatibility condition, recall that we also have

$$\varepsilon' = \varepsilon \circ F\beta \circ \alpha U'.$$

Using the naturality of β, then α, then β, then ε, then β, then α, and finally the counit compatibility, the following computation concludes:

$$\begin{aligned} U\varepsilon F \circ (U\alpha \circ \beta F')UF \circ U'F'(U\alpha \circ \beta F') &= U\varepsilon F \circ U\alpha UF \circ \beta F'UF \circ U'F'U\alpha \circ U'F'\beta F' \\ &= U\varepsilon F \circ \beta FUF \circ U'\alpha UF \circ U'F'U\alpha \circ U'F'\beta F' \\ &= U\varepsilon F \circ \beta FUF \circ U'FU\alpha \circ U'\alpha UF' \circ U'F'\beta F' \\ &= \beta F \circ U'\varepsilon F \circ U'FU\alpha \circ U'\alpha UF' \circ U'F'\beta F' \\ &= \beta F \circ U'\alpha \circ U'\varepsilon F' \circ U'\alpha UF' \circ U'F'\beta F' \\ &= U\alpha \circ \beta F' \circ U'\varepsilon F' \circ U'\alpha UF' \circ U'F'\beta F' \\ &= U\alpha \circ \beta F' \circ U'\varepsilon F' \circ U'F\beta F' \circ U'\alpha U'F' \\ &= U\alpha \circ \beta F' \circ U'\varepsilon' F'. \end{aligned}$$

To prove functoriality of $\mathcal{M}^{\mathrm{co}}$, the identity axiom is obvious; for a composable pair of morphisms

$$(F, U, \eta, \varepsilon) \xrightarrow{(\alpha,\beta)} (F', U', \eta', \varepsilon') \xrightarrow{(\alpha',\beta')} (F'', U'', \eta'', \varepsilon'')$$

in $(\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}})^{\mathrm{op}}$, we compute

$$\begin{aligned} \mathcal{M}^{\mathrm{co}}((\alpha', \beta') \circ (\alpha, \beta)) &= \mathcal{M}^{\mathrm{co}}((\alpha \circ \alpha', \beta \circ \beta')) \\ &= (\mathrm{id}_{\mathcal{C}}, U(\alpha \circ \alpha') \circ (\beta \circ \beta')F'') \\ &= (\mathrm{id}_{\mathcal{C}}, U\alpha \circ U\alpha' \circ \beta F'' \circ \beta' F'') \\ &= (\mathrm{id}_{\mathcal{C}}, U\alpha \circ \beta F' \circ U'\alpha' \circ \beta' F'') \\ &= (\mathrm{id}_{\mathcal{C}}, U'\alpha' \circ \beta' F'') \circ (\mathrm{id}_{\mathcal{C}}, U\alpha \circ \beta F') \\ &= \mathcal{M}^{\mathrm{co}}((\alpha', \beta')) \circ \mathcal{M}^{\mathrm{co}}((\alpha, \beta)). \end{aligned}$$

□

We now also obtain an arrow from the category of arrows to the category of (large) categories.

Proposition 3.25. *There is a functor*

$$L : \mathbf{Monad}^{\mathrm{co}} \to \mathbf{CAT}.$$

Proof. Define the action of L on objects by carrying a monad $(\mathcal{C}, \mathbf{T} = (T, \eta, \mu))$ to the Eilenberg–Moore category $\mathcal{C}^{\mathbf{T}}$, and a morphism

$$(F : \mathcal{C} \to \mathcal{C}', \alpha : T'F \Rightarrow FT) : (\mathcal{C}, \mathbf{T}) \to (\mathcal{C}', \mathbf{T}')$$

to the functor

$$L((F, \alpha)) : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}'^{\mathbf{T}'}$$

that sends a $\mathbf{T}$-algebra $(A, a : TA \to A)$ to the $\mathbf{T}'$-algebra

$$(FA, Fa \circ \alpha_A : T'FA \to FA)$$

and a morphism

$$f : (A, a) \to (B, b) \in \mathcal{C}^{\mathbf{T}}$$

to the $\mathbf{T}'$-algebra homomorphism

$$Ff : (FA, Fa \circ \alpha_A) \to (FB, Fb \circ \alpha_B).$$

To see it is well-defined, observe first that $(FA, Fa \circ \alpha_A : T'FA \to FA)$ is a $\mathbf{T}'$-algebra. Indeed, using the unit compatibility condition for (F, α) and then the unit axiom for (A, a), we have

$$\begin{aligned} Fa \circ \alpha_A \circ \eta'_{FA} &= Fa \circ F\eta_A \\ &= F(a \circ \eta_A) \\ &= \mathrm{id}_{FA}. \end{aligned}$$

Moreover, using naturality of α in the first step, the multiplication axiom for (A, a) in the second, and the multiplication compatibility condition for (F, α) in the third, we obtain

$$\begin{aligned} Fa \circ \alpha_A \circ T'Fa \circ T'\alpha_A &= Fa \circ FTa \circ \alpha_{TA} \circ T'\alpha_A \\ &= Fa \circ F\mu_A \circ \alpha_{TA} \circ T'\alpha_A \\ &= Fa \circ \alpha_A \circ \mu'_{FA}. \end{aligned}$$

Thus $(FA, Fa \circ \alpha_A)$ is indeed a $\mathbf{T}'$-algebra.

Furthermore, Ff defines a $\mathbf{T}'$-algebra homomorphism. Indeed, using that f is a $\mathbf{T}$-algebra homomorphism in the first step and naturality of α in the second, we have

$$\begin{aligned} Ff \circ Fa \circ \alpha_A &= Fb \circ FTf \circ \alpha_A \\ &= Fb \circ \alpha_B \circ T'Ff. \end{aligned}$$

Functoriality of $L((F, \alpha))$ directly follows from that of F.

Finally, we prove functoriality of L. The identity axiom is immediate:

$$L((\mathrm{id}_{\mathcal{C}}, \mathrm{id}_T)) = \mathrm{id}_{\mathcal{C}^{\mathbf{T}}}.$$

For a composable pair of morphisms

$$(\mathcal{C}, \mathbf{T}) \xrightarrow{(F,\alpha)} (\mathcal{C}', \mathbf{T}') \xrightarrow{(F',\alpha')} (\mathcal{C}'', \mathbf{T}''),$$

recall that

$$(F', \alpha') \circ (F, \alpha) = (F'F, F'\alpha \circ \alpha' F).$$

For a $\mathbf{T}$-algebra (A, a), we have

$$\begin{aligned} L((F', \alpha') \circ (F, \alpha))(A, a) &= \big(F'FA,\ F'Fa \circ F'\alpha_A \circ \alpha'_{FA}\big) \\ &= L((F', \alpha'))\,(FA,\ Fa \circ \alpha_A) \\ &= \big(L((F', \alpha')) \circ L((F, \alpha))\big)(A, a). \end{aligned}$$

Moreover, on a morphism f,

$$\begin{aligned} L((F', \alpha') \circ (F, \alpha))(f) &= F'Ff \\ &= \big(L((F', \alpha')) \circ L((F, \alpha))\big)(f). \end{aligned}$$

Hence

$$L((F', \alpha') \circ (F, \alpha)) = L((F', \alpha')) \circ L((F, \alpha)),$$

which concludes the proof. □

Remark 3.26. In particular, we obtain a composite functor $(\mathbf{Adj}_{\mathcal{C}}^{\mathcal{D}})^{\mathrm{op}} \xrightarrow{\mathcal{M}^{\mathrm{co}}} \mathbf{Monad}^{\mathrm{co}} \xrightarrow{L} \mathbf{CAT}$.

Finally, we may assemble the previous results to prove Corollary 3.27.

Corollary 3.27. *If a functor $U : \mathcal{D} \to \mathcal{C}$ admits two left adjoints $F, F' : \mathcal{U} \to \mathcal{D}$, then there is an isomorphism of categories $\mathcal{I} : \mathcal{C}^{\mathbf{T}'} \xrightarrow{\cong} \mathcal{C}^{\mathbf{T}}$ between the induced Eilenberg-Moore categories, which fits into a commutative triangle:*

$$\begin{array}{ccc} \mathcal{C}^{\mathbf{T}} & \xleftarrow[\cong]{\mathcal{I}} & \mathcal{C}^{\mathbf{T}'} \\ & {\scriptstyle K} \nwarrow \quad \nearrow {\scriptstyle K'} & \\ & D & \end{array} .$$

Proof. By Remark 3.23, we have an isomorphism $(\varepsilon D' \circ F\eta', \mathrm{id}_U) : (F, U, \eta, \varepsilon) \xrightarrow{\cong} (F', U, \eta', \varepsilon')$ in $\mathbf{Adj}^{\mathcal{D}}_{\mathcal{C}}$. Now, the crucial categorical fact is that functors preserve isomorphisms by Lemma 2.24 (this was the motivation behind all of the previous constructions!). Applying the functors in the Remark 3.26, we obtain an isomorphism of categories $\mathcal{I} : \mathcal{C}^{\mathbf{T}'} \xrightarrow{\cong} \mathcal{C}^{\mathbf{T}}$; explicitly, a $\mathbf{T}'$-algebra $(A, a : T'A \to A)$ is mapped to the $\mathbf{T}$-algebra $(A, a \circ U\varepsilon_{F'A} \circ UF\eta'_A)$, and a $\mathbf{T}'$-algebra homomorphism f is mapped to itself.

Now, recalling the definitions of the canonical functors K and K' of Proposition 3.13, we compute the action on objects:

$$\begin{aligned} (\mathcal{I} \circ K)D &= \mathcal{I}((UD, U\varepsilon'_D)) \\ &= (UD,\, U\varepsilon'_D \circ U\varepsilon_{F'UD} \circ UF\eta'_{UD}) \\ &= (UD,\, U(\varepsilon'_D \circ \varepsilon_{F'UD} \circ F\eta'_{UD})) \\ &= (UD,\, U(\varepsilon_D \circ F(U\varepsilon'_D \circ \eta'_{UD}))) \\ &= (UD, U\varepsilon_D) \\ &= K'D. \end{aligned}$$

where we used naturality of ε followed by a triangle identity for the adjunction $F' \dashv U$. On morphisms:

$$(\mathcal{I} \circ K)g = \mathcal{I}(Ug) = Ug = K'g,$$

as desired. □

3.4 Canonical presentations via free algebras

We now explore how the notion of a group presentation from group theory can be generalized to the category of algebras on a monad. Moreover, we develop the necessary language to prove Theorem 3.41 in section 3.5.

We begin with a remark from group theory.

Remark 3.28. Recall from Example 3.4 (ii) the free-forgetful adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Group}$$

and the underlying monad $\mathbf{T}$ on $\mathbf{Set}$. By the isomorphism (2.3), any group G can be identified with the quotient of the free group on the underlying set of G, namely FUG, by the smallest normal subgroup that, given any word of words, identifies the evaluation of each subword inside the group G, with the concatenation (after potentially applying inversion to a subword) of the subwords, both processes outputting an element of FUG. Using the explicit description of coequalizers in **Group** from Example 2.69 (iii), we discover that *any* group G is the coequalizer of the following diagram in **Group**:

$$FUFUG \underset{\varepsilon_{FUG}}{\overset{FU\varepsilon_G}{\rightrightarrows}} FUG \xrightarrow{\varepsilon_G} G. \tag{3.2}$$

Since $U : \mathbf{Group} \to \mathbf{Set}$ is strictly monadic by Example 3.14, any group G corresponds to the $\mathbf{T}$-algebra $(UG, U\varepsilon_G)$, and a group homomorphism $\varphi : G \to G'$ to the $\mathbf{T}$-algebra homomorphism $U\varphi$. Under this isomorphism of categories, the coequalizer diagram (3.2) translates into the following coequalizer diagram in the category of algebras $\mathbf{Set}^{\mathbf{T}}$:

$$(UFUFUG, U\varepsilon_{FUFUG}) \underset{U\varepsilon_{FUG}}{\overset{UFU\varepsilon_G}{\rightrightarrows}} (UFUG, U\varepsilon_{FUG}) \xrightarrow{U\varepsilon_G} (UG, \varepsilon_G).$$

In the more abstract framework of a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$, one could thus suspect that *any* $\mathbf{T}$-algebra admits a *canonical presentation*, expressed as a "quotient" of a free $\mathbf{T}$-algebra.

Theorem 3.29. *Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. Any $\mathbf{T}$-algebra $(A, TA \xrightarrow{a} A)$ fits into a coequalizer diagram in $\mathcal{C}^{\mathbf{T}}$:*

$$(T^2A, \mu_{TA}) \underset{\mu_A}{\overset{Ta}{\rightrightarrows}} (TA, \mu_A) \xrightarrow{a} (A, a) \tag{3.3}$$

We first present a direct proof.

Proof. First, we observe that all arrows in the diagram (3.3) are indeed $\mathbf{T}$-homomorphisms: Ta by naturality of μ, μ_A by the monad associativity axiom, and a because it is a structure map, which is also the reason why $a : (TA, \mu_A) \to (A, a)$ defines a cocone under the parallel pair.

Let (B, b) be a $\mathbf{T}$-algebra and $h : (TA, \mu_A) \to (B, b)$ a $\mathbf{T}$-algebra homomorphism so that $h \circ Ta = h \circ \mu_A$. If there exists $k : (A, a) \to (B, b)$ so that the bottom triangle commutes

$$\begin{array}{ccccc} (T^2A, \mu_{TA}) & \underset{\mu_A}{\overset{Ta}{\rightrightarrows}} & (TA, \mu_A) & \xrightarrow{q} & (A, a) \\ & & & \searrow^{h} & \downarrow{\scriptstyle k} \\ & & & & (B, b) \end{array},$$

then we see that k is uniquely determined:

$$k \circ a = h \implies k = k \circ a \circ \eta_A = h\eta_A,$$

where we used the triangle identity defining the $\mathbf{T}$-algebra (A, a). It remains to show that $k = h \circ \eta_A$ is a $\mathbf{T}$-homomorphism $(A, a) \to (B, b)$. That is, we need to show that the following square commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Th \circ T\eta_A} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[h \circ \eta_A]{} & B \end{array}.$$

Using that h is a $\mathbf{T}$-homomorphism, as well as the monad unit axiom, we compute the top-right composite

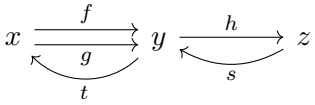
$$b \circ Th \circ T\eta_A = h \circ \mu_A \circ T\eta_A = h.$$

The bottom-right composite again uses the $\mathbf{T}$-algebra triangle identity of (A, a):

$$h \circ \eta_A \circ a = h.$$

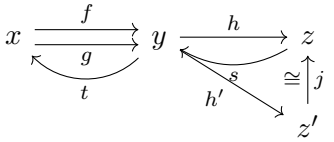
Therefore both composites are equal, which concludes the proof. □

Definition 3.30. A *split coequalizer diagram* in a category $\mathcal{C}$ consists of objects and morphisms as displayed



so that $h \circ f = h \circ g$, $h \circ s = \mathrm{id}_z$, $g \circ t = \mathrm{id}_y$ and $f \circ t = s \circ h$.

Remark 3.31. Suppose we have a cocone under the parallel pair $f, g : x \rightrightarrows y$ that is isomorphic to the fork underlying a split coequalizer diagram:

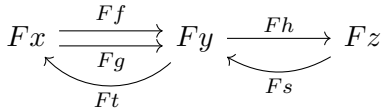


with $j : z' \xrightarrow{\cong} z$ so that $j \circ h' = h$. Then we may replace the "decorators" t and s by new decorators t' and s', defined by $t' = t$ and $s' = s \circ j$, so as to obtain a new split coequalizer diagram. Indeed, we verify $h' \circ f = h' \circ g$ (which holds by definition of a cocone under the parallel pair), and $j \circ h' \circ s' = j \circ h' \circ s \circ j = h \circ s \circ j = j$ implies that $h' \circ s' = \mathrm{id}_{z'}$, $g \circ t' = g \circ t = \mathrm{id}_y$ and $s' \circ h' = s \circ j \circ h' = s \circ h = f \circ t = f \circ t'$, as desired.

Proposition 3.32. *The underlying fork of a split coequalizer diagram is a coequalizer. Moreover, it is an absolute coequalizer, i.e., it is preserved by any functor.*

Proof. Given $k : y \to w$ so that $k \circ f = k \circ g$, if there is $l : z \to w$ such that $k = l \circ h$, then it is necessarily given by $l = l \circ h \circ s = k \circ s$. Conversely, the triangle commutes because $k = k \circ g \circ t = k \circ f \circ t = k \circ s \circ h$.

Applying a functor $F : \mathcal{C} \to \mathcal{D}$ to the split coequalizer diagram yields a diagram

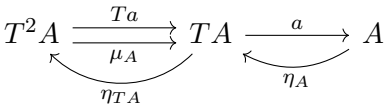


(3.4)

so that $Fh \circ Ff = Fh \circ Fg$, $Fh \circ s = \mathrm{id}_{Fz}$, $Fg \circ Ft = \mathrm{id}_{Fy}$ and $Ff \circ Ft = Fs \circ Fh$ by functoriality of F. In particular, it defines a split coequalizer diagram in $\mathcal{D}$, so by what we have just proven, the underlying fork of the transformed diagram, which is the transformed fork underlying the split coequalizer in $\mathcal{C}$, defines a coequalizer in $\mathcal{D}$, as desired. □

Remark 3.33. In particular, for any functor $F : \mathcal{C} \to \mathcal{D}$, Fh is an epimorphism, in the diagram (3.4).

Example 3.34. Using the monad identities as well as **T**-algebra properties, for any **T**-algebra $(A, a : TA \to A)$, the diagram

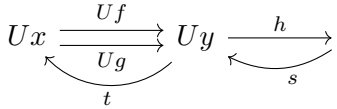


(3.5)

is a split coequalizer diagram in $\mathcal{C}$.

Definition 3.35. Let $U : \mathcal{D} \to \mathcal{C}$ be a functor.

- A *U-split pair* is a parallel pair $f, g : x \rightrightarrows y$ in $\mathcal{D}$ so that after applying U, the transformed parallel pair can be completed into a split coequalizer diagram in $\mathcal{C}$, called a *U-split coequalizer*:



- *U creates coequalizers of U-split pairs* if any U-split pair $f, g : x \rightrightarrows y$ admits a coequalizer fork

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \xrightarrow{h'} z'$$

whose image under U is isomorphic (in the category of cocones) to the fork underlying the U-split coequalizer in $\mathcal{C}$:

$$\begin{array}{ccccc} Ux & \underset{Ug}{\overset{Uf}{\rightrightarrows}} & Uy & \xrightarrow{h} & z \\ & & & \searrow^{Uh'} & \cong \downarrow \\ & & & & Uz' \end{array} .$$

Moreover, any fork in $\mathcal{D}$ with this property should be a coequalizer of the parallel pair f, g.

- *U strictly creates coequalizers of U-split pairs* if the U-split coequalizer of any U-split pair admits a unique lift to a fork in $\mathcal{D}$, which is required to be a coequalizer:

$$\begin{array}{ccccc} x & \underset{g}{\overset{f}{\rightrightarrows}} & y & \overset{\exists! h'}{\cdots\!\!\twoheadrightarrow} & z' \\ & & \downarrow U & & \\ Ux & \underset{Ug}{\overset{Uf}{\rightrightarrows}} & Uy & \xrightarrow{h} & z \end{array} .$$

Remark 3.36. A functor $U : \mathcal{D} \to \mathcal{C}$ that strictly creates coequalizers of U-split pairs also creates coequalizers of U-split pairs.

Proposition 3.37. *For any monad $\boldsymbol{T} = (T, \eta, \mu)$ on $\mathcal{C}$, the forgetful functor $U^{\boldsymbol{T}} : \mathcal{C}^{\boldsymbol{T}} \to \mathcal{C}$ strictly creates coequalizers of $U^{\boldsymbol{T}}$-split pairs.*

Proof. Consider a parallel pair $f, g : (A, a) \to (B, b)$ in $\mathcal{C}^{\mathbf{T}}$ that is split by $U^{\mathbf{T}}$:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{s}{\overset{h}{\rightleftarrows}} C \quad (t : B \to A) .$$

We need to show that C admits a unique $\mathbf{T}$-algebra structure (C, c) so that h defines a $\mathbf{T}$-homomorphism $h : (B, b) \to (C, c)$ and that the fork

$$(A, a) \underset{g}{\overset{f}{\rightrightarrows}} (B, b) \xrightarrow{h} (C, c) \tag{3.6}$$

defines a coequalizer diagram in $\mathcal{C}^{\mathbf{T}}$.

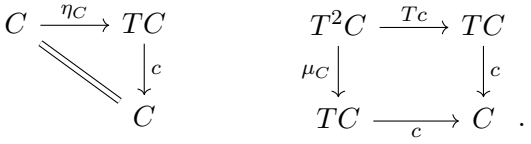
Suppose C admitted a $\mathbf{T}$-algebra structure. By Proposition 3.32, the functor $T : \mathcal{C} \to \mathcal{C}$ preserves split coequalizers; moreover, the map $h \circ b : TB \to C$ defines a cocone under the parallel pair $Tf, Tg : TA \rightrightarrows TB$, witnessed by

$$h \circ b \circ Tf = h \circ f \circ a = h \circ g \circ a = h \circ b \circ Tg.$$

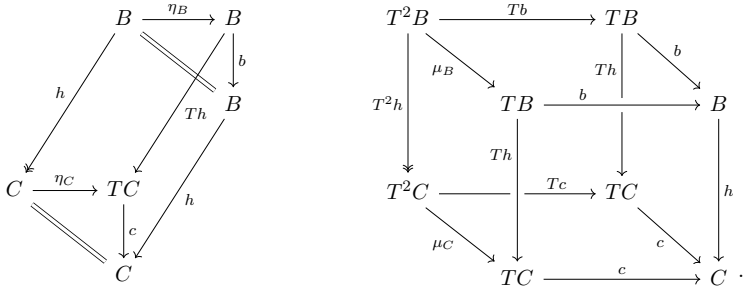
By the universal property of the coequalizer Th, we see that $c : TC \to C$ must be the unique map so that the right square commutes :

$$\begin{array}{ccccc} TA & \underset{Tg}{\overset{Tf}{\rightrightarrows}} & TB & \xrightarrow{Th} & TC \\ a \downarrow & & b \downarrow & & \exists! c \downarrow \\ A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow[h]{} & C \end{array} .$$

Next, we prove that (C, c) is a $\mathbf{T}$-algebra, from which it will follow by definition of c that $h : (B, b) \to (C, c)$ is indeed a $\mathbf{T}$-homomorphism. That is, we need to check that the following diagrams commute:



Consider the following diagrams:



Let us focus on proving the triangle identity. Each face of the prism is known to commute apart from the desired bottom triangle. From a diagram chase we see that

$$c \circ \eta_C \circ h = c \circ Th \circ \eta_B = h \circ b \circ \eta_B = h.$$

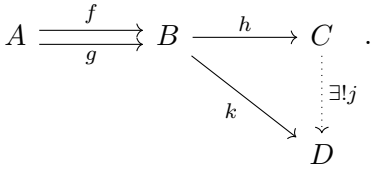
Since h is an epimorphism, we conclude that the bottom triangle commutes.

To prove the square identity, observe that each face of the cube is known to commute, except from the bottom square. Again, a diagram chase implies that

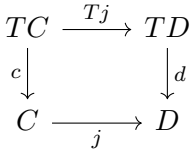
$$c \circ \mu_C \circ T^2 h = c \circ Th \circ \mu_B = h \circ b \circ \mu_B = h \circ b \circ Th = c \circ Th \circ Tb = c \circ Tc \circ T^2 h.$$

Since by Remark 3.33 $T^2 h$ is an epimorphism, the desired square commutes.

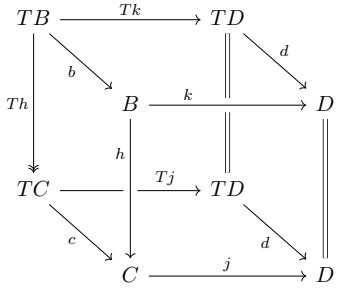
Finally, we need to prove that the uniquely lifted fork (3.6) is a coequalizer in $\mathcal{C}^{\mathbf{T}}$. Consider a map $k : (B, b) \to (D, d)$ so that $k \circ f = k \circ g$. By the universal property of the coequalizer $h : B \twoheadrightarrow C$ in $\mathcal{C}$, there is a unique $j : C \to D$ in $\mathcal{C}$ so that $k = j \circ h$:



Thus it suffices to prove that j lifts to a $\mathbf{T}$-homomorphism in $\mathcal{C}^{\mathbf{T}}$; that is, to show that the following square commutes:



The following cube will be useful:



All faces apart from the bottom one are known to commute; since Th is an epimorphism, a diagram chase enables us to conclude. □

We now obtain another proof of Theorem 3.29.

Proof. By (3.5), the fork

$$(T^2A, \mu_{TA}) \underset{\mu_A}{\overset{Ta}{\rightrightarrows}} (TA, \mu_A)$$

is a $U^{\mathbf{T}}$-split pair. Proposition 3.37 now allows us to conclude. □

We can now generalize our coequalizer (3.2) in **Group**.

Corollary 3.38. *Let*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D}$$

be a monadic adjunction. For any object $D \in \mathcal{D}$, there is a coequalizer diagram:

$$FUFUD \underset{\varepsilon_{FUD}}{\overset{FU\varepsilon_D}{\rightrightarrows}} FUD \xrightarrow{\varepsilon_D} D \tag{3.7}$$

Proof. Since $F \dashv U$ is monadic, the canonical comparison functor $K : \mathcal{D} \xrightarrow{\simeq} \mathcal{C}^{\mathbf{T}}$ is an equivalence. Applying K to the fork (3.7), and applying Theorem 3.29 to the $\mathbf{T}$-algebra $KD = (UD, U\varepsilon_D)$, we conclude that

$$(UFUFUD, U\varepsilon_{FUFUD}) \underset{U\varepsilon_{FUD}}{\overset{UFU\varepsilon_D}{\rightrightarrows}} (UFUD, U\varepsilon_{FUD}) \xrightarrow{U\varepsilon_D} (UD, U\varepsilon_D)$$

is a coequalizer diagram in $\mathcal{C}^{\mathbf{T}}$. Since by Lemma 2.79 equivalences reflect colimits, the conclusion follows. □

3.5 Monadic functors

We now study functors that behave like the forgetful functor $U :$ **Group** $\to$ **Set**, as seen in Example 3.14.

Definition 3.39. An adjunction

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D}$$

is *monadic* (resp. *strictly monadic*) if the canonical comparison functor $K : \mathcal{D} \xrightarrow{\simeq} \mathcal{C}^{\mathbf{T}}$ from Proposition 3.13 is an equivalence (resp. isomorphism). A functor $U : \mathcal{D} \to \mathcal{C}$ is *monadic* (resp. *strictly monadic*) if it admits a left adjoint $F \dashv U$ defining a *monadic* (resp. *strictly monadic*) adjunction.

Remark 3.40. By Corollary 3.27, the monadicity or strict monadicity property of a functor $U : \mathcal{D} \to \mathcal{C}$ is *independant* of the choice of left adjoint F, whenever it exists, since post-compostion by an isomorphism of categories $\mathcal{I}$ does not change the properties of a functor (in this case, the notion of equivalence or isomorphism). For this reason, one may hope that the concept of monadicity of a functor $U : \mathcal{D} \to \mathcal{C}$ could be rephrased as a special property satisfied by U, an intrinsic property, with no mention of a particular left adjoint F. This is precisely the content of Beck's famous monadicity theorem; see [6].

Theorem 3.41. *(monadicity theorem) An adjunction*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$$

is monadic if and only if U creates coequalizers of U-split pairs.

Here we provide a substantially more detailed proof than the compact one presented in [1].

Proof. $\Rightarrow$: By definition, the canonical comparison functor $K : D \xrightarrow{\simeq} \mathcal{C}^{\mathbf{T}}$ is an equivalence. Recall that K fits into a commutative triangle, which will guide us in the course of the proof:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow[\simeq]{K} & \mathcal{C}^T \\ & {\scriptstyle U} \searrow \quad \swarrow {\scriptstyle U^{\mathbf{T}}} & \\ & \mathcal{C} & \end{array} \quad . \tag{3.8}$$

To demonstrate the first property of creation of coequalizers of U-split pairs in Definition 3.35, consider a U-split pair $f, g : A \rightrightarrows B$ in $\mathcal{D}$, where the image of the parallel pair under U can be upgraded to a split coequalizer diagram in $\mathcal{C}$:

$$UA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UB \xrightarrow{h} C \quad . \tag{3.9}$$
$$t \qquad\qquad s$$

By the triangle (3.8), $Kf, Kg : KA \rightrightarrows KB$ is a U^T-split pair in $\mathcal{C}^{\mathbf{T}}$. Proposition 3.37 lifts the underyling fork in $\mathcal{C}$ to a coequalizer diagram in $\mathcal{C}^{\mathbf{T}}$:

$$KA \underset{Kg}{\overset{Kf}{\rightrightarrows}} KB \xrightarrow{h} (C, c) \quad .$$

Since K is an equivalence, it creates colimits by Lemma 2.79, so we obtain a coequalizer

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h'} C' \tag{3.10}$$

in $\mathcal{C}$, which is preserved by K. In particular, by uniqueness of colimits (Proposition 2.65), there is a unique isomorphism which makes the bottom-right triangle commute:

$$\begin{array}{ccccc} KA & \underset{Kg}{\overset{Kf}{\rightrightarrows}} & KB & \xrightarrow{h} & (C, c) \\ & & & {\scriptstyle Kh'} \searrow & \downarrow \cong \\ & & & & KC' \end{array} \quad .$$

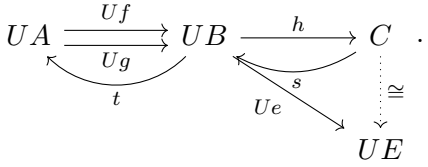
Applying $U^{\mathbf{T}}$, we see that the image of the constructed coequalizer (3.10) in $\mathcal{D}$ under $U = U^{\mathbf{T}}K$ is indeed isomorphic to the fork underlying the U-split coequalizer diagram (3.9) in $\mathcal{C}$:

$$\begin{array}{ccccc} UA & \underset{Ug}{\overset{Uf}{\rightrightarrows}} & UB & \xrightarrow{h} & C \\ & & & {\scriptstyle Uh'} \searrow & \downarrow \cong \\ & & & & UC' \end{array} \quad .$$

We now verify the second condition of creation of U-split coequalizers. Consider a fork in $\mathcal{D}$

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{e} E$$

whose image under U is isomorphic to the underlying fork in the U-split coequalizer diagram (3.9):



By Remark 3.31, we obtain a new split coequalizer diagram, which is the image under $U^{\mathbf{T}}$ of the fork

$$KA \underset{Kg}{\overset{Kf}{\rightrightarrows}} KB \xrightarrow{Ke} KE$$

in $\mathcal{C}^{\mathbf{T}}$. By Proposition 3.37, this fork is a coequalizer diagram in $\mathcal{C}^{\mathbf{T}}$. Since K is an equivalence, Lemma 2.79 implies that K reflects colimits, from which we deduce that $e : B \twoheadrightarrow E$ is a coequalizer, as desired.

$\Leftarrow$: Assume that U creates coequalizers of U-split pairs. We will construct an inverse equivalence $L : \mathcal{C}^{\mathbf{T}} \to \mathcal{D}$ to K.

Motivation for the definition of L. We will first give an intuitive and non-rigorous reason as to why the action on objects of L is defined the way it is, before diving into the technical details. If K were an equivalence, then up to isomorphism any $\mathbf{T}$-algebra (A, a) corresponds to an element $D \in \mathcal{D}$, i.e., $(A, a) \cong KD$. We wish to recover D from the $\mathbf{T}$-algebra; recall that by Corollary 3.38 we have a coequalizer diagram in $\mathcal{D}$

$$FUFUD \underset{\varepsilon_{FUD}}{\overset{FU\varepsilon_D}{\rightrightarrows}} FUD \xrightarrow{\varepsilon_D} D \ .$$

Replacing $KD = (UD, U\varepsilon_D)$ by (A, a), we see that D is the coequalizer of the parallel pair $Fa, \varepsilon_{FA} : FUFA \to FA$, involving only the $\mathbf{T}$-algebra and no mention of D.

Rigorous construction of L. For a $\mathbf{T}$-algebra (A, a), define $L(A, a)$ to be the coequalizer

$$FUFA \underset{\varepsilon_{FA}}{\overset{Fa}{\rightrightarrows}} FA \longrightarrow\!\!\!\!\!\rightarrow L(A, a) \ . \tag{3.11}$$

The coequalizer (3.11) exists in $\mathcal{D}$; indeed, the parallel pair $Fa, \varepsilon_{FA} : FUFA \to FA$ is U-split by Example 3.34, and U creates coequalizers of U-split pairs by hypotheses. Moreover, its image under U is isomorphic to the underlying fork of

$$T^2A \underset{\eta_{TA}}{\overset{Ta}{\underset{\mu_A}{\rightrightarrows}}} TA \underset{\eta_A}{\overset{a}{\leftrightarrows}} A \ .$$

For a $\mathbf{T}$-algebra homomorphism $f : (A, a) \to (B, b)$, note that the two left squares below commute (the top route by definiton of a $\mathbf{T}$-algebra homomorphism, the bottom route by naturality of ε). Define $Lf : L(A, a) \to L(B, b)$ to be the unique map so that the right square commutes:

$$\begin{array}{ccccc} FUFA & \underset{\varepsilon_{FA}}{\overset{Fa}{\rightrightarrows}} & FA & \longrightarrow\!\!\!\!\!\rightarrow & L(A,a) \\ {\scriptstyle FUFf}\downarrow & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle \exists!\, Lf} \\ FUFB & \underset{\varepsilon_{FB}}{\overset{Fb}{\rightrightarrows}} & FB & \longrightarrow\!\!\!\!\!\rightarrow & L(B,b) \end{array} \ .$$

Functoriality of L follows from the functoriality of F, vertical commutative square diagram pasting, and the uniqueness property of the image under L of a $\mathbf{T}$-algebra homomorphism.

We now prove that $KL \cong \mathrm{id}_{\mathcal{C}^{\mathbf{T}}}$. As mentionned previously, the parallel pair $Fa, \varepsilon_{FA} : FUFA \to FA$ is U-split; since $U = U^{\mathbf{T}}K$ and $U^{\mathbf{T}}$ strictly creates coequalizers of $U^{\mathbf{T}}$-split pairs by Proposition 3.37, we conclude that K preserves the coequalizer (3.11), which by Proposition 2.65 results in an isomorphism making the bottom-right triangle commute:

$$\begin{array}{ccccc} (T^2A, \mu_{TA}) & \overset{Ta}{\underset{\mu_A}{\rightrightarrows}} & (TA, \mu_A) & \xrightarrow{\quad a \quad} & (A, a) \\ & & & \searrow & \downarrow \cong \\ & & & & KL(A,a) \ . \end{array}$$

We now prove naturality. Given a $\mathbf{T}$-homomorphism $f : (A, a) \to (B, b)$, we wish to demonstrate that the following square commutes:

$$\begin{array}{ccc} (A,a) & \xrightarrow{\cong} & KL(A,a) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle KLf} \\ (B,b) & \xrightarrow{\cong} & KL(B,b) \ . \end{array}$$

Since (A, a) is a coequalizer (in particular, a colimit), it suffices, by Lemma 2.71 to prove that the two composites are equal when pre-composed with the coequalizer leg $a : (TA, \mu_A) \to (A, a)$. Upon applying K to the square

$$\begin{array}{ccc} FA & \longrightarrow & L(A,a) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Lf} \\ FB & \longrightarrow & L(B,b) \end{array}$$

we obtain the following commutative square:

$$\begin{array}{ccc} (TA, \mu_A) & \longrightarrow & KL(A,a) \\ {\scriptstyle KFf=Tf}\downarrow & & \downarrow{\scriptstyle KLf} \\ (TB, \mu_B) & \longrightarrow & KL(B,b) \ . \end{array}$$

Now, the equality of our desired 3-morphism composites follows from the commutativity of the outer rectangle we have just established, as well as the commutativity of the left square (because $f : (A, a) \to (B, b)$ is a $\mathbf{T}$-homomorphism):

$$\begin{array}{ccccc} (TA, \mu_A) & \xrightarrow{a} & (A,a) & \xrightarrow{\cong} & KL(A,a) \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle KLf} \\ (TB, \mu_B) & \xrightarrow[b]{} & (B,b) & \xrightarrow{\cong} & KL(B,b) \ . \end{array}$$

Finally, we prove that $LK \cong \mathrm{id}_{\mathcal{D}}$. For $D \in \mathcal{D}$, we have by Corollary 3.38 two coequalizer diagrams, resulting in an isomorphism making the bottom-right triangle commute:

$$\begin{array}{ccccc} FUFUD & \overset{FU\varepsilon_D}{\underset{\varepsilon_{FUD}}{\rightrightarrows}} & FUD & \longrightarrow & LKD \\ & & & {\scriptstyle \varepsilon_D}\searrow & \downarrow \cong \\ & & & & D \end{array}$$

To prove naturality, we adopt the same strategy. For $g : D \to D' \in \mathcal{D}$, we want to show that the naturality square

$$\begin{array}{ccc} LKD & \xrightarrow{\cong} & D \\ {\scriptstyle LKg}\downarrow & & \downarrow{\scriptstyle g} \\ LKD' & \xrightarrow{\cong} & D' \end{array}$$

commutes. To do so, it suffices to prove the two composites coincide when pre-composing by $FUD \twoheadrightarrow LKD$. By definition of the action of L on morphisms, the following square commutes:

$$\begin{array}{ccc} FUD & \twoheadrightarrow & LKD \\ {\scriptstyle FUg}\downarrow & & \downarrow{\scriptstyle LKg} \\ FUD' & \twoheadrightarrow & LKD' \end{array} .$$

Finally, by naturality of ε, the outer rectangle commutes, which proves that the 3-morphism composites are equal:

$$\begin{array}{ccccc} & & \overset{\varepsilon_D}{\frown} & & \\ FUD & \twoheadrightarrow & LKD & \xrightarrow{\cong} & D \\ {\scriptstyle FUg}\downarrow & & \downarrow{\scriptstyle LKg} & & \downarrow{\scriptstyle g} \\ FUD' & \twoheadrightarrow & LKD' & \xrightarrow{\cong} & D' \\ & & \underset{\varepsilon_{D'}}{\smile} & & \end{array} .$$

This concludes the proof that L defines an inverse equivalence to K, thus proving that K is an equivalence. □

Remark 3.42. Notice that we now have another proof of the independance of the chosen left adjoint in Definition 3.39. Indeed, if $U : \mathcal{D} \to \mathcal{C}$ admits two left adjoints $F, F' : \mathcal{C} \to \mathcal{D}$, by Theorem 3.41:

$$F \dashv U \text{ is monadic} \iff U \text{ creates coequalizers of } U\text{-split pairs} \iff F' \dashv U \text{ is monadic.}$$

There is also a strict version of this theorem, whose proof is very similar, which we will simply state here.

Theorem 3.43. *(strict version of the monadicity theorem) An adjunction*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{D}$$

is strictly monadic if and only if U strictly creates coequalizers of U-split pairs.

Our next goal is to prove that the forgeful functor $U : \mathbf{cHaus} \to \mathbf{Set}$ is (strictly) monadic, in order to illustrate an application of the monadicity theorem. We will prove that it strictly creates coequalizers of U-split pairs, and will not prove here that it admits a left adjoint (an explicit construction is given by the Stone–Čech compactification of a discrete set; see [7]). Before diving into the proof, we first give an equivalent definition of a topological space.

Proposition 3.44. *There is a one-to-one correspondance been topological spaces $(X, \mathcal{T})$ and pairs $(X, \overline{(-)})$, where $\overline{(-)} : \mathcal{P}X \to \mathcal{P}X$, called the Kuratowski closure operator, satisfies the following four axioms for any $A, B \in \mathcal{P}(X)$.*

(i) $\overline{\emptyset} = \emptyset$

(ii) $A \subseteq \overline{A}$

(iii) $\overline{\overline{A}} = \overline{A}$

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. Given a space $(X, \mathcal{T})$, we define the usual closure of a subspace

$$\overline{A} = \bigcap_{\substack{A \subseteq C \\ C \subseteq X,\, C \text{ closed}}} C.$$

This closure map indeed satisfies the four axioms.

Conversely, given a pair $(X, \overline{(-)})$, let $\mathcal{T} := \{A \subset X \mid \overline{A^c} = A^c\}$. To show that this defines a topology on X, first note that $\overline{(-)}$ is monotone; indeed, for $A \subset B$ we have

$$\overline{A} \subset \overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{B},$$

so in particular, for a collection of subsets $(C_i)_{i \in I}$,

$$\overline{\cap_{i \in I} C_i} \subset \cap_{i \in I} \overline{C_i}.$$

Now the axioms for a topology follow.

(i) $X \subset \overline{X} \subset X \implies \emptyset \in \mathcal{T}$, and $\overline{\emptyset} = \emptyset \implies X \in \mathcal{T}$;

(ii) if $A, B \in \mathcal{T}$, then $\overline{(A \cap B)^c} = \overline{A^c \cup B^c} = \overline{A^c} \cup \overline{B^c} = A^c \cup B^c = (A \cap B)^c \implies A \cap B \in \mathcal{T}$;

(iii) if $(A_i)_{i \in I} \subset \mathcal{T}$, we have by the observation above that $\overline{(\cup_{i \in I} A_i)^c} = \overline{\cap_{i \in I} A_i^c} \subset \cap_{i \in I} \overline{A_i^c} = \cap_{i \in I} A_i^c = (\cup_{i \in I} A_i)^c \implies \cup_{i \in I} A_i \in \mathcal{T}$.

The fact that the constructions are inverses crucially uses axiom (iii) for the closure operator. □

Proposition 3.45. *The forgetful functor* $U : \mathbf{cHaus} \to \mathbf{Set}$ *is strictly monadic*

Proof. Applying the strict version of the monadicity theorem (Theorem 3.43), it suffices to show that U strictly creates coequalizers of U-split pairs. We will use the language of the closure operator to work with topological spaces and continous maps. For a function $f : X \to Y$ between topological spaces, it is known that f is continous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$, and closed (meaning continous and preserving closed subspaces) if and only if $f(\overline{A}) = \overline{f(A)}$ for all $A \subset X$. As any continous map between compact Hausdorff spaces is closed, we may use the latter condition to express continuity.

Now, consider a parallel pair $f, g : X \rightrightarrows Y \in \mathbf{cHaus}$ which is split by U, that is, we have a split coequalizer diagram in **Set**:

$$X \underset{t}{\overset{f}{\underset{g}{\rightrightarrows}}} Y \underset{s}{\overset{h}{\rightleftarrows}} Z \ .$$

We need to show that Z lifs uniquely to a compact Hausdorff space so that h is continous. The covariant power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, like any functor, preserves the split coequalizer by Proposition 3.32, and hence we see that the closure operator on Z is the unique map making the right square commute:

$$\begin{array}{ccccc} \mathcal{P}X & \underset{g_*}{\overset{f_*}{\rightrightarrows}} & \mathcal{P}Y & \xrightarrow{h_*} & \mathcal{P}Z \\ {\scriptstyle \overline{(-)}^X}\downarrow & & \downarrow{\scriptstyle \overline{(-)}^Y} & & \vdots{\scriptstyle \exists! \, \overline{(-)}^Z} \\ \mathcal{P}X & \underset{g_*}{\overset{f_*}{\rightrightarrows}} & \mathcal{P}Y & \xrightarrow{h_*} & \mathcal{P}Z \end{array} \ .$$

We prove that $(Z, \overline{(-)}^Z)$ satisfies the four axioms of previous Prop for all $C, D \in \mathcal{P}Z$, using the section $s_* : \mathcal{P}Z \to \mathcal{P}Y$ to h_* and the same properties for $(Y, \overline{(-)}^Y)$.

(i) $\overline{\emptyset}^Z = \overline{h_*(\emptyset)}^Z = h_*(\overline{\emptyset}^Y) = h_*(\emptyset) = \emptyset$

(ii) $C = h_*(s_*(C)) \subset h_*(\overline{s_*(C)}^Y) = \overline{h_*(s_*(C))}^Z = \overline{C}^Z$

(iii) $\overline{\overline{C}^Z}^Z = h_*(\overline{\overline{s_*(C)}^Y}^Y) = h_*(\overline{s_*(C)}^Y) = \overline{C}^Z$

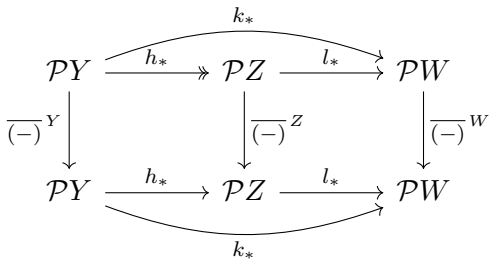
(iv) $\overline{C \cup D}^Z = h_*\Big(\overline{s_*(C \cup D)}^Y\Big) = h_*\Big(\overline{s_*(C)}^Y\Big) \cup h_*\Big(\overline{s_*(D)}^Y\Big) = \overline{C}^Z \cup \overline{D}^Z$

We have thus lifted Z to a topological space; we still need to prove it is a compact Hausdorff space. Since Y is compact, and $h : Y \twoheadrightarrow Z$ is surjective, Z is compact. To prove that Z is Hausdorff, note that since h is closed and surjective, each point in Z is closed. Take two distinct points $z, z' \in Z$; as they are closed and h continous, the fibers $h^{-1}(z)$, $h^{-1}(z')$ are closed and disjoint in Y. As compact Haudorff spaces are normal, the fibers can be separated by open disjoint sets so that $U \supset h^{-1}(z)$, $U' \supset h^{-1}(z')$. Then $V := h(U^c)^c$, $V' := h(U'^c)^c$ are open in Z, disjoint because $V \cap V' = (h(U^c) \cap h(U'^c))^c = h(U^c \cup U'^c)^c = h((U \cap U')^c)^c = \emptyset$, and contain z and z' respectively.

It remains to show that

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

is a coequalizer diagram in **cHaus**. Let $k : Y \to W \in$ **cHaus** so that $k \circ f = k \circ g$. Applying U and using the universal property of the coequalizer $h : Y \twoheadrightarrow Z$ in **Set**, there exists a unique $l : Z \to W$ so that $k = l \circ h$. To prove that l is continous (equivalently, closed), note that the outer rectangle commutes by continuity of k; the right square then commutes because h_* is an epimorphism:



This proves that $l : Z \to W$ lifts to define a map in **cHaus**, as desired. □

Example 3.46. In Example 3.14, we have shown that $U :$ **Group** $\to$ **Set** is strictly monadic by explicitly examining whether the canonical comparison functor K is an isomorphism. Other examples of strictly monadic functors are the forgetful functors from **Monoid**, ${}_R$**Mod**, **Vect**$_k$, **Ab** and **Ring** to **Set**.

The following lemma describes a key property of monadic functors.

Lemma 3.47. *A monadic functor $U : A \to \mathcal{C}$ reflects isomorphisms.*

Proof. Recall that $U = U^{\mathbf{T}}K$, where the canonical comparison functor $K : A \xrightarrow{\simeq} \mathcal{C}^{\mathbf{T}}$ is an equivalence. Proposition 3.9 implies that $U^{\mathbf{T}}$ reflects isomorphisms. Since equivalences reflect isomorphisms by Lemma 2.79, the proof is complete. □

A nice application of this result is the following characterization of homeomorphisms of compact Hausdorff spaces.

Corollary 3.48. *Any bijective continuous function between compact Hausdorff spaces is a homeomorphism.*

In group theory courses, the standard definition of a group isomorphism is a bijective group homomorphism, which is not the categorical definition of an isomorphism in a category. It does, however, turn out to be equivalent by Lemma 2.24 and Lemma 3.47, or by directly verifying that the inverse is again a group homomorphism. This alternative characterization does not hold for topological spaces. Indeed, the categorical definition requires the inverse map to be continuous, which is not always the case - for instance, if a set X admits two topologies so that $\mathcal{T} \subsetneq \mathcal{T}'$, then $\mathrm{id}_X : (X, \mathcal{T}') \to (X, \mathcal{T})$ is a continuous bijection whose inverse is not continuous. By the contrapositive of Lemma 3.47, we have just proven the following proposition.

Proposition 3.49. *The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is not monadic.*

3.6 Limits and colimits in the category of algebras

The theorem below explores the connection between monadic functors and creation of limits/colimits.

Theorem 3.50. *Let*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{A}$$

be a (strictly) monadic adjunction. Then $U : \mathcal{A} \to \mathcal{C}$ (strictly) creates:

(i) any limits that $\mathcal{C}$ has;

(ii) any colimits that $\mathcal{C}$ has and that T and T^2 preserve, where $T = UF$.

Proof. Note that since a composite of functors that strictly create limits/colimits again strictly creates them, likewise for creation, that isomorphisms strictly create limits/colimits, and equivalence create limits/colimits, it suffices to prove that $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ strictly creates (i) and (ii). Here, $\mathbf{T}$ is the monad underlying the adjunction $F \dashv U$.

In the course of the proof, we will adopt the following notation. If $\mathbf{D} : J \to \mathcal{C}^{\mathbf{T}}$ is a diagram, then for $j \in J$, we write $\mathbf{D}j = (Dj, \delta_j : TDj \to Dj)$. For a morphism $f : j \to j' \in J$, we write $Df : Dj \to Dj'$ for the underlying map in $\mathcal{C}$ of the $\mathbf{T}$-algebra homomorphism $\mathbf{D}f$.

To prove (i), consider a diagram $\mathbf{D} : J \to \mathcal{C}^{\mathbf{T}}$ so that $U^{\mathbf{T}}\mathbf{D} : J \to \mathcal{C}$ has a limit cone $\lambda : \Delta L \Rightarrow U^{\mathbf{T}}\mathbf{D}$; its legs are morphisms $\lambda_j : L \to Dj$ for each $j \in J$.

Since for all $f : j \to j' \in J$, we have a commutative square

$$\begin{array}{ccc} TDj & \xrightarrow{TDf} & TDj' \\ {\scriptstyle \delta_j}\downarrow & & \downarrow{\scriptstyle \delta'_j} \\ Dj & \xrightarrow[Df]{} & Dj' \end{array},$$

we obtain a natural transformation $\delta : TU^{\mathbf{T}}\mathbf{D} \Rightarrow U^{\mathbf{T}}\mathbf{D}$, and hence an induced cone

$$\Delta TL \overset{T\lambda}{\Longrightarrow} TU^T\mathbf{D} \overset{\delta}{\Rightarrow} U^T\mathbf{D}.$$

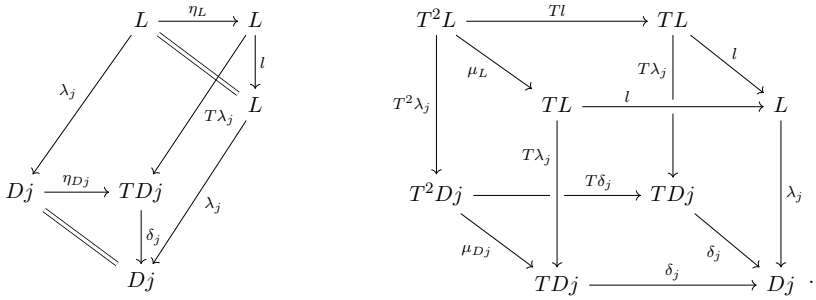
in $\mathcal{C}$. By the universal property of the limit cone λ, there exists a unique $l : TL \to L$ so that

$$\begin{array}{ccc} TL & \xrightarrow{T\lambda_j} & TDj \\ {\scriptstyle \exists! l}\vdots & & \downarrow{\scriptstyle \delta'_j} \\ L & \xrightarrow[\lambda_j]{} & Dj \end{array}$$

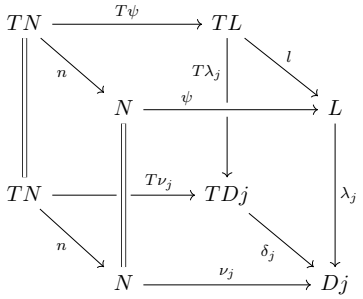
commutes. To conclude that λ admits a unique lift to a cone over $\mathbf{D}$ in $\mathcal{C}^{\mathbf{T}}$, it remains to prove that (L, l) defines a $\mathbf{T}$-algebra, i.e., to check that the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\eta_L} & TL \\ & \searrow^{=} & \downarrow{\scriptstyle l} \\ & & L \end{array} \qquad \begin{array}{ccc} T^2L & \xrightarrow{Tl} & TL \\ {\scriptstyle \mu_L}\downarrow & & \downarrow{\scriptstyle l} \\ TL & \xrightarrow[l]{} & L \end{array} .$$

Since L is a limit, by Lemma 2.71, it suffices to prove that the two composite are equal when post-composed with λ_j, for each $j \in J$. In the diagrams below, all faces but the top ones are known to commute; as seen in other proofs, a diagram chase then enables us to conclude:



Finally, to prove that the lifted cone $\tilde{\lambda} : \Delta(L, l) \Rightarrow \mathbf{D}$ is a limit cone, take another cone $\nu : \Delta(N, n) \Rightarrow \mathbf{D}$; then $U^{\mathbf{T}}\nu : \Delta N \Rightarrow U^{\mathbf{T}}\mathbf{D}$ defines a cone over $U^{\mathbf{T}}\mathbf{D}$. Since λ is a limit cone, there exists a unique $\psi : N \to L$ such that $U^{\mathbf{T}}\nu = \lambda \circ \Delta\psi$. It remains to prove that ψ defines a $\mathbf{T}$-homomorphism $(N, n) \to (L, l)$; all faces apart from the top one are known to commute, so since the two composites in the square of interest are equal, by a diagram chase, after post-composing by λ_j, we are done.



For (ii), assume that $\mathbf{D} : J \to \mathcal{C}^{\mathbf{T}}$ is a diagram with colimit cocone $\lambda : U^{\mathbf{T}}\mathbf{D} \Rightarrow \Delta L$ in $\mathcal{C}$, so that both $T\lambda : TU^{\mathbf{T}}\mathbf{D} \Rightarrow \Delta TL$ and $T^2\lambda : T^2U^{\mathbf{T}}\mathbf{D} \Rightarrow \Delta T^2L$ are also colimit cocones. If λ were to lift to $\tilde{\lambda} : \mathbf{D} \Rightarrow \Delta(L, l)$, then as $\delta : TU^{\mathbf{T}}\mathbf{D} \Rightarrow U^{\mathbf{T}}\mathbf{D}$ induces a cocone

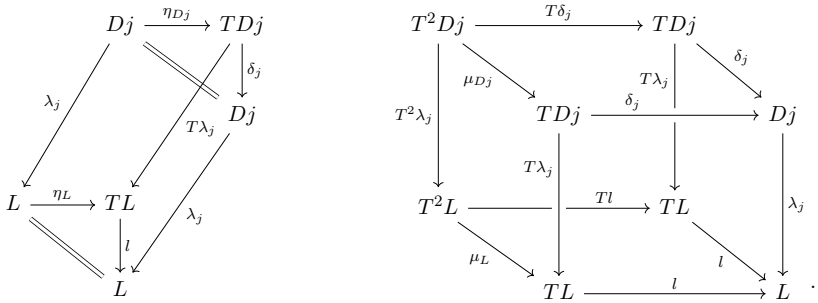
$$TU^{\mathbf{T}}\mathbf{D} \overset{\delta}{\Rightarrow} U^{\mathbf{T}}\mathbf{D} \overset{\lambda}{\Rightarrow} \Delta L$$

in $\mathcal{C}$ over $TU^{\mathbf{T}}\mathbf{D}$, and $T\lambda$ is a colimit cone, $l : TL \to L$ is the unique map so that

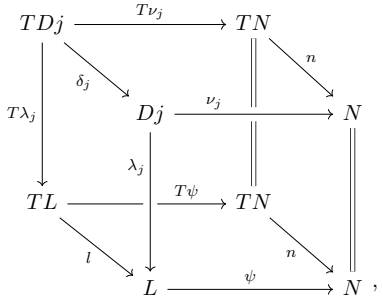
$$\begin{array}{ccc} TDj & \xrightarrow{T\lambda_j} & TL \\ {\scriptstyle \delta_j}\downarrow & & \vdots\,\exists! l \\ Dj & \xrightarrow[\lambda_j]{} & L \end{array}$$

commutes $\forall j \in J$. We need to show that l is a $\mathbf{T}$-algebra structure map. As before, the following

diagrams are useful, where this time we use the universal property of λ and $T^2\lambda$, respectively:



Finally, to show that $\tilde{\lambda}$ is a colimit cocone in $\mathcal{C}^{\mathbf{T}}$ over $\mathbf{D}$, take another cocone $\nu : \mathbf{D} \Rightarrow \Delta(N, n)$. By the universal property of λ, there exists a unique $\psi : L \to N$ so that $U^T\nu = \lambda \circ \Delta\psi$. To show that ψ lifts to a $\mathbf{T}$-homomorphism, we use the universal property of $T\lambda$



which concludes the proof. □

Since **Set** is complete, we obtain a nice corollary.

Corollary 3.51. *Any category $\mathcal{A}$ that is monadic over* **Set** *(i.e., there is a monadic functor $U : \mathcal{A} \to$ **Set**) is complete.*

Before presenting the final big result about monads, we give a motivation from group theory. We have seen in Proposition 2.97 how to compute free products (the categorical coproduct in **Group**) of groups. Here, we explore another approach. Given groups G, G', we would like the coproduct $G * G'$ to consist of formal finite words constructed using elements of $UG \sqcup UG'$, so that any given word which contains a string of elements of the same group - say, G - should be identified with the word obtained by evaluating that string inside G. After some thought, this implies in formulas that the coproduct of the two groups should be the following coequalizer in **Group**:

$$F(UFUG \sqcup UFUG') \xrightarrow{F(U\varepsilon_G \sqcup U\varepsilon'_G)} F(UG \sqcup UG') \twoheadrightarrow G * G' \ ,$$
$$F\kappa \searrow \qquad \nearrow \varepsilon_{F(UG \sqcup UG')}$$
$$FUF(UG \sqcup UG')$$

where $UFUG \sqcup UFUG' \xrightarrow{\kappa} UF(UG \sqcup UG')$ is the canonical comparison map (2.2) from the coproduct of the images of UG, UG' under the functor UF to the functor UF applied to the coproduct $UG \sqcup UG'$. In particular, coproducts in **Group** can be computed using coequalizers! As we have already used in Remark 3.28, since $U :$ **Group** $\to$ **Set** is strictly monadic, a group G corresponds to the $\mathbf{T}$-algebra $(A, a) := (UG, U\varepsilon_G)$. Translating this diagram in the isomorphic category of $\mathbf{T}$-algebras, we obtain the following coequalizer diagram in $\mathbf{Set}^{\mathbf{T}}$:

$$(T(TA \sqcup TA'), \mu_{TA \sqcup TA'}) \xrightarrow{T(a \sqcup a')} (T(A \sqcup A'), \mu_{A \sqcup A'}) \twoheadrightarrow (A, a) \sqcup (A', a') \ .$$
$$T\kappa \searrow \qquad \nearrow \mu_{A \sqcup A'}$$
$$(T^2(A \sqcup A'), \mu_{T(A \sqcup A')})$$

Generalizing to the more abstract framework of monads, one might suspect that the existence of coproducts in $\mathcal{C}$ as well as coequalizers in $\mathcal{C}^{\mathbf{T}}$ suffices to prove the existence of coproducts in $\mathcal{C}^{\mathbf{T}}$.

These observations lead us to the following theorem.

Theorem 3.52. *Let $U : \mathcal{A} \to \mathcal{C}$ be a monadic functor so that $\mathcal{C}$ admits all small coproducts. If $\mathcal{A}$ has coequalizers, then $\mathcal{A}$ is cocomplete.*

Here we provide all the details of the proof, while [1] only provides the coequalizer diagram (without showing it is a coproduct).

Proof. By Theorem 2.82, it suffices to show that $\mathcal{A}$ admits coproducts. Since $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ is an equivalence, and equivalences create colimits by Lemma 2.79, it suffices to prove that $\mathcal{C}^{\mathbf{T}}$ has coproducts.

Consider an I-indexed family of $\mathbf{T}$-algebras $\{(A_i, a_i) \mid i \in I\}$. Recall the comparison map (2.2) $\kappa : \coprod_{i\in I} TA_i \to T(\coprod_{i\in I} A_i)$ with the property that for each $i \in I$, the following triangle commutes:

$$\begin{array}{ccc} TA_i & & \\ \downarrow & \searrow^{T(A_i \to \coprod_{i'\in I} A_{i'})} & \\ \coprod_{i'\in I} TA_{i'} & \xrightarrow[\kappa]{} & T\left(\coprod_{i'\in I} A_{i'}\right) . \end{array}$$

Take the coequalizer:

$$\begin{array}{ccccc} (T(\coprod_{i\in I} TA_i), \mu_{\coprod_{i\in I} TA_i}) & \xrightarrow{T(\coprod_{i\in I} a_i)} & (T(\coprod_{i\in I} A_i), \mu_{\coprod_{i\in I} A_i}) & \xrightarrow{q} & (A, a) . \\ & \searrow^{T\kappa} \quad \nearrow_{\mu_{\coprod_{i\in I} A_i}} & & & \\ & (T^2(\coprod_{i\in I} A_i), \mu_{T(\coprod_{i\in I} A_i)}) & & & \end{array} \tag{3.12}$$

We show that $(A, a) \cong \coprod_{i\in I}(A_i, a_i)$. The first step is to define the coproduct legs. For $i \in I$, define $\lambda_i : (A_i, a_i) \to (A, a)$ to be the composite

$$A_i \xrightarrow{\eta_{A_i}} TA_i \to T(\coprod_{i\in I} A_i) \xrightarrow{q} A.$$

We prove that it defines a $\mathbf{T}$-homomorphism.

$$\begin{array}{ccccccc} TA_i & \xrightarrow{T\eta_{A_i}} & T^2A_i & \longrightarrow & T^2(\coprod_{i'\in I} A_{i'}) & \xrightarrow{Tq} & TA \\ \downarrow{\scriptstyle a_i} & & & & & & \downarrow{\scriptstyle a} \\ A_i & \xrightarrow[\eta_{A_i}]{} & TA_i & \longrightarrow & T(\coprod_{i'\in I} A_{i'}) & \xrightarrow[q]{} & A \end{array}$$

The top-right composite equals

$$\begin{aligned} a \circ Tq \circ T^2\left(A_i \to \coprod_{i'\in I} A_{i'}\right) \circ T\eta_{A_i} &= q \circ \mu_{\coprod_{i'\in I} A_{i'}} \circ T^2\left(A_i \to \coprod_{i'\in I} A_{i'}\right) \circ T\eta_{A_i} \\ &= q \circ T\left(A_i \to \coprod_{i'\in I} A_{i'}\right) \circ \mu_{A_i} \circ T\eta_{A_i} \\ &= q \circ T\left(A_i \to \coprod_{i'\in I} A_{i'}\right) : TA_i \to A, \end{aligned}$$

using that q is a $\mathbf{T}$-homomorphism, μ is natural, and the monad unit axiom.

For the bottom-left composite, we will use the fact that the coequalizer q is a cocone under the parallel pair in (3.12):

$$q \circ \mu_{\coprod_{i' \in I} A_{i'}} \circ T\kappa = q \circ T(\coprod_{i' \in I} a_{i'})$$

Precomposing this equality by $TA_i \to \coprod_{i' \in I} TA_{i'} \xrightarrow{\eta_{\coprod_{i' \in I} TA_{i'}}} T(\coprod_{i' \in I} TA_{i'})$, we obtain on the one hand

$$\begin{aligned} q \circ \mu_{\coprod_{i' \in I} A_{i'}} \circ T\kappa \circ \eta_{\coprod_{i' \in I} TA_{i'}} \circ \left(TA_i \to \coprod_{i' \in I} TA_{i'} \right) &= q \circ \mu_{\coprod_{i' \in I} A_{i'}} \circ \eta_{T(\coprod_{i' \in I} A_{i'})} \circ \kappa \circ \left(TA_i \to \coprod_{i' \in I} TA_{i'} \right) \\ &= q \circ T \left(A_i \to \coprod_{i' \in I} A_{i'} \right), \end{aligned}$$

by naturality of η, the other unit axiom of a monad, and the triangle identity of κ. On the other hand, this is the same as

$$\begin{aligned} q \circ T\left(\coprod_{i' \in I} a_{i'} \right) \circ \eta_{\coprod_{i' \in I} TA_{i'}} \circ \left(TA_i \to \coprod_{i' \in I} TA_{i'} \right) &= q \circ T\left(\coprod_{i' \in I} a_{i'} \right) \circ T\left(TA_i \to \coprod_{i' \in I} TA_{i'} \right) \circ \eta_{TA_i} \\ &= q \circ T\left(A_i \to \coprod_{i' \in I} A_{i'} \right) \circ Ta_i \circ \eta_{TA_i} \\ &= q \circ T\left(A_i \to \coprod_{i' \in I} A_{i'} \right) \circ \eta_{A_i} \circ a_i, \end{aligned}$$

thanks to the naturality of η, definition of $\coprod_{i' \in I} a_{i'}$ and again naturality of η, which is exactly the bottom-right composite! Thus $\lambda_i : (A_i, a_i) \to (A, a)$ is a $\mathbf{T}$-homomorphism.

We now show that the universal property of the coproduct holds. Consider a cocone $(\nu_i : (A_i, a_i) \to (B, b))_{i \in I}$. Define

$$p := T(\coprod_{i' \in I} A_{i'}) \xrightarrow{T(\coprod_{i' \in I} \nu_{i'})} TB \xrightarrow{b} B.$$

Then $p : (T(\coprod_{i' \in I} A_{i'}, \mu_{\coprod_{i' \in I} A_{i'}}) \to (B, b)$ defines a $\mathbf{T}$-homomorphism because

$$\begin{array}{ccccc} T^2(\coprod_{i' \in I} A_{i'}) & \xrightarrow{T(\coprod_{i' \in I} \nu_{i'})} & T^2 B & \xrightarrow{Tb} & TB \\ {\scriptstyle \mu_{\coprod_{i' \in I} A_{i'}}} \downarrow & & \downarrow {\scriptstyle \mu_B} & & \downarrow {\scriptstyle b} \\ T(\coprod_{i' \in I} A_{i'}) & \xrightarrow[T(\coprod_{i' \in I} \nu_{i'})]{} & TB & \xrightarrow[b]{} & B \end{array}$$

commutes by naturality of μ and the associativity law for the $\mathbf{T}$-algebra (B, b). Moreover, p

defines a cocone under the parallel pair in (3.12) because

$$\begin{aligned} b \circ T\left(\coprod_{i' \in I} \nu_{i'}\right) \circ \mu_{\coprod_{i' \in I} A_{i'}} \circ T\kappa &= b \circ \mu_B \circ T^2\left(\coprod_{i' \in I} \nu_{i'}\right) \circ T\kappa \\ &= b \circ \mu_B \circ T\left(\coprod_{i' \in I} T\nu_{i'}\right) \\ &= b \circ T\left(b \circ \coprod_{i' \in I} T\nu_{i'}\right) \\ &= b \circ T\left(\coprod_{i' \in I} \nu_{i'} \circ a_{i'}\right) \\ &= b \circ T\left(\coprod_{i' \in I} \nu_{i'}\right) \circ T\left(\coprod_{i' \in I} a_{i'}\right), \end{aligned}$$

by naturality of μ, the κ-triangle, the associativity law of (B, b) and the fact that ν_i defines a $\mathbf{T}$-homomorphism. Now, by the universal property of the coequalizer $(T(\coprod_{i' \in I} A_{i'}, \mu_{\coprod_{i' \in I} A_{i'}}) \overset{q}{\twoheadrightarrow} (A, a)$, there is a unique $j : (A, a) \to (B, b)$ so that $p = j \circ q$.

We now show that the equation $p = j \circ q$ is equivalent to commutation of j with the legs; that is, $j \circ \lambda_i = \nu_i$.

If $p = j \circ q$, observe that

$$\begin{aligned} j \circ \lambda_i &= j \circ q \circ \left(TA_i \to T\left(\coprod_{i' \in I} A_{i'}\right)\right) \circ \eta_{A_i} \\ &= p \circ \left(TA_i \to T\left(\coprod_{i' \in I} A_{i'}\right)\right) \circ \eta_{A_i} \\ &= b \circ T\nu_i \circ \eta_{A_i} \\ &= b \circ \eta_B \circ \nu_i \\ &= \nu_i, \end{aligned}$$

using the definitions, naturality of ν and the $\mathbf{T}$-algebra triangle identity for (B, b).

Conversely, if $j : (A, a) \to (B, b)$ commutes with the legs, then $p = j \circ q$; indeed, notice that the domain of p is the free $\mathbf{T}$-algebra $(T(\coprod_{i' \in I} A_{i'}, \mu_{\coprod_{i' \in I} A_{i'}})$. This means that we can use transposition with respect to the adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$ to determine if these maps are equal. On the one hand, transposing p yields

$$p \circ \eta_{\coprod_{i' \in I} A_{i'}} = b \circ T(\coprod_{i' \in I} \nu_{i'}) \circ \eta_{\coprod_{i' \in I} A_{i'}} = b \circ \eta_B \circ \coprod_{i' \in I} \nu_{i'} = \coprod_{i' \in I} \nu_{i'},$$

by naturality of η and the $\mathbf{T}$-algebra triangle identity for (B, b). On the other hand, note that for all $i \in I$,

$$A_i \to \coprod_{i' \in I} A_{i'} \xrightarrow{\eta_{\coprod_{i' \in I} A_{i'}}} T(\coprod_{i' \in I} A_{i'}) \overset{q}{\to} A \overset{j}{\to} B = j \circ \lambda_i = \nu_i,$$

and thus the transpose of $j \circ q$ is

$$j \circ q \circ \eta_{\coprod_{i' \in I} A_{i'}} = \coprod_{i' \in I} \nu_{i'}.$$

Therefore the transposes are equal, which implies that $p = j \circ q$.

Having proven that the two conditions are equivalent, since there is a unique j satisfying the first, then that same j is the unique map satisfying the second condition. This is exactly the statement that $(\lambda_i : (A_i, a_i) \to (A, a))_{i \in I}$ defines a coproduct cocone, as desired. □

4 Monoidal Categories

We would like to categorify the notion of a monoid, which is a set equipped with a binary operation and identity element; the law is required to be associative, and the identity unital. Upgrading sets and functions to the categorical world of categories and functors, and, importantly, relaxing the notion of equality to that of an isomorphism, leads us to a categorification of a monoid.

Definition 4.1. A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, where

- $\mathcal{C}$ is a category;
- $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the *tensor product bifunctor*;
- $\mathbf{1} \in \mathcal{C}$ is the unit;
- a is a natural isomorphism, called *associator*, whose domain and codomain functors are displayed below:

$$\begin{array}{ccc} (\mathcal{C} \times \mathcal{C}) \times \mathcal{C} & \xrightarrow{\mathcal{C} \times - \otimes -} & \mathcal{C} \times \mathcal{C} \\ {\scriptstyle - \otimes - \times \mathcal{C}}\downarrow & \overset{a}{\underset{\cong}{\swarrow}} & \downarrow{\scriptstyle - \otimes -} \\ \mathcal{C} \times \mathcal{C} & \xrightarrow[- \otimes -]{} & \mathcal{C} \end{array};$$

- $l : \mathbf{1} \otimes - \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}$ is the *left unitor*;
- $r : - \otimes \mathbf{1} \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}$ is the *right unitor*;

which satisfy the following two coherence axioms.

- *The Pentagon axiom.* For any objects $W, X, Y, Z \in \mathcal{C}$, the following pentagon commutes:

$$\begin{array}{ccc} & ((W \otimes X) \otimes Y) \otimes Z & \\ {\scriptstyle a_{W \otimes X, Y, Z}}\swarrow & & \searrow{\scriptstyle a_{W,X,Y} \otimes \mathrm{id}_Z} \\ (W \otimes X) \otimes (Y \otimes Z) & & (W \otimes (X \otimes Y)) \otimes Z \\ {\scriptstyle a_{W,X,Y \otimes Z}}\downarrow & & \downarrow{\scriptstyle a_{W, X \otimes Y, Z}} \\ W \otimes (X \otimes (Y \otimes Z)) & \xleftarrow[\mathrm{id}_W \otimes a_{X,Y,Z}]{} & W \otimes ((X \otimes Y) \otimes Z) \end{array}. \tag{4.1}$$

- *The Triangle axiom.* For any objects $X, Y \in \mathcal{C}$, the following triangle commutes:

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\quad a_{X,\mathbf{1},Y} \quad} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array}. \tag{4.2}$$

Remark 4.2. By Proposition 2.46, since $\mathrm{id}_{\mathcal{C}}$ is an equivalence, we discover that $\mathbf{1} \otimes -$ and $- \otimes \mathbf{1}$ define equivalences $\mathcal{C} \xrightarrow{\simeq} \mathcal{C}$.

Remark 4.3. If $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a monoidal category and $\mathcal{D} \subset \mathcal{C}$ is a full subcategory so that

- $A, B \in \mathcal{D} \implies A \otimes B \in \mathcal{D}$, and
- $\mathbf{1} \in \mathcal{D}$,

then $\mathcal{D}$ inherits the monoidal category structure of $\mathcal{C}$ by restricting each component to $\mathcal{D}$.

Remark 4.4. Given a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, there are two natural ways to obtain new monoidal categories.

The first one is the *opposite monoidal category* $(\mathcal{C}^{op}, \otimes^{op}, \mathbf{1}, a^{op}, l^{op}, r^{op})$, where $- \otimes^{op} - : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$ is the opposite functor introduced in some Remark 2.22. The components of the new associator and unitors are defined by

$$a^{op}_{X,Y,Z} := (a^{-1}_{X,Y,Z})^{op} : (X \otimes^{op} Y) \otimes^{op} Z \to X \otimes^{op} (Y \otimes^{op} Z) \in \mathcal{C}^{op},$$

$$l^{op}_X := l^{-1}_X : \mathbf{1} \otimes^{op} X \to X \in \mathcal{C}^{op},$$

and

$$r^{op}_X := r^{-1}_X : X \otimes^{op} \mathbf{1} \to X \in \mathcal{C}^{op}$$

for all $X, Y, Z \in \mathcal{C}^{op}$.

The second construction is the *reverse monoidal category* $(\mathcal{C}, \otimes^{rev}, \mathbf{1}, a^{rev}, l^{rev}, r^{rev})$. Let us denote by $\sigma : \mathcal{C} \times \mathcal{C} \xrightarrow{\cong} \mathcal{C} \times \mathcal{C}$ the functor which swaps the two variables - that is, $\sigma(X, Y) = (Y, X)$ on objects - and by $\tau : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{\cong} \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ the functor which reverses the order of the three variables - i.e. $\tau(X, Y, Z) = (Z, Y, X)$. Then $- \otimes^{rev} - := - \otimes - \circ \sigma$; in other words,

$$X \otimes^{rev} Y := Y \otimes X$$

for any $X, Y \in \mathcal{C}$ and likewise for morphisms. The associator $a^{rev} := (a\tau)^{-1}$ is defined by whiskering and inverting; its component at $X, Y, Z \in \mathcal{C}$ is

$$a^{rev}_{X,Y,Z} := a^{-1}_{Z,Y,X} : (X \otimes^{rev} Y) \otimes^{rev} Z \to X \otimes^{rev} (Y \otimes^{rev} Z).$$

Finally, the unitors are exchanged: $l^{rev} := r$ and $r^{rev} := l$.

Proposition 4.5. *Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. Then we have two additional commutative triangles for any objects $X, Y \in \mathcal{C}$:*

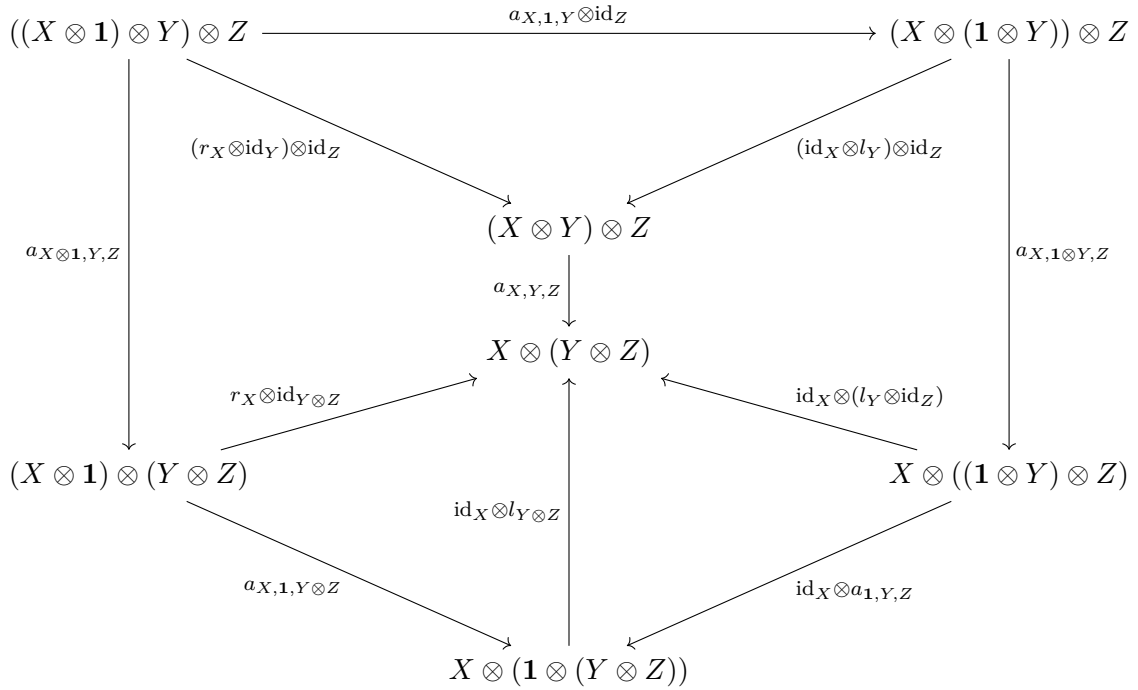
$$\begin{array}{ccc} (\mathbf{1} \otimes X) \otimes Y & \xrightarrow{\quad a_{\mathbf{1},X,Y} \quad} & \mathbf{1} \otimes (X \otimes Y) \\ & {\scriptstyle l_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle l_{X \otimes Y}} & \\ & X \otimes Y & \end{array} \tag{4.3}$$

and

$$\begin{array}{ccc} (X \otimes Y) \otimes \mathbf{1} & \xrightarrow{\quad a_{X,Y,\mathbf{1}} \quad} & X \otimes (Y \otimes \mathbf{1}) \\ & {\scriptstyle r_{X \otimes Y}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes r_Y} & \\ & X \otimes Y & \end{array} \tag{4.4}$$

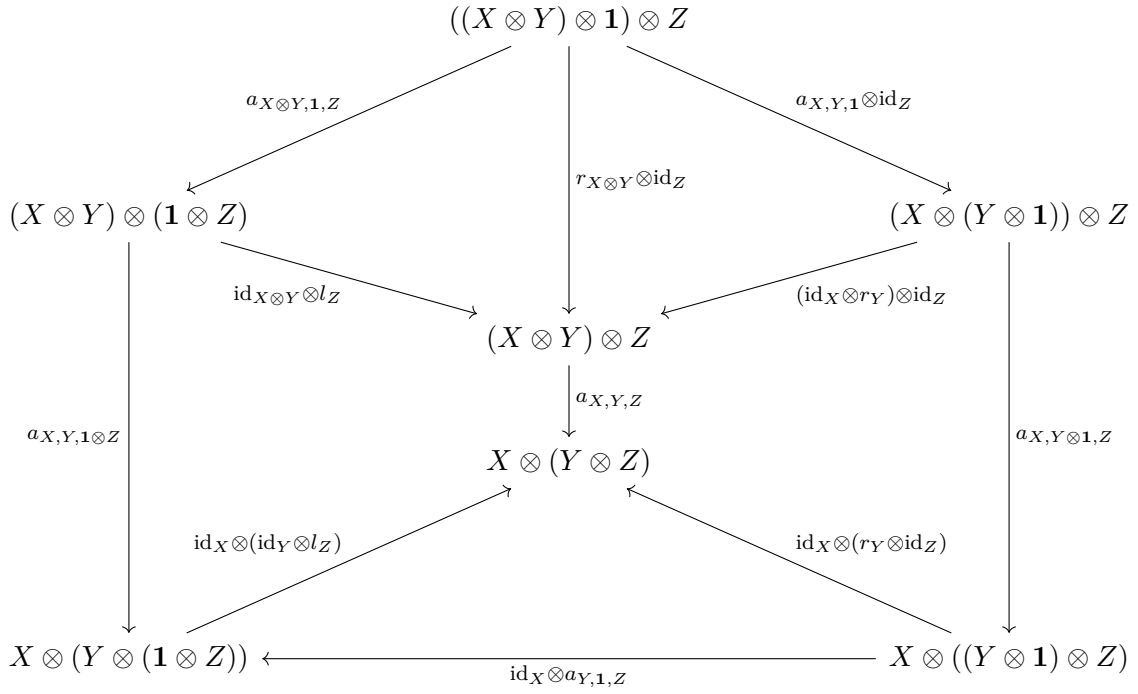
.

Proof. We first prove the triangle identity (4.3). For objects $X, Y, Z \in \mathcal{C}$, consider the following diagram:



The outer pentagon commutes by the pentagon axiom (4.1); the upper triangle and lower-left triangle commute by the triangle axiom (4.2). Moreover, the left and right quadrilaterals commute by naturality of a. Since all the morphisms involved are isomorphisms, a diagram chase implies that the bottom-right triangle commutes. Setting $X = \mathbf{1}$, the faithfulness of $\mathbf{1} \otimes -$ implies the commutativity of the desired triangle (4.3).

To prove the triangle identity (4.4), consider the diagram



for any objects $X, Y, Z \in \mathcal{C}$. The outer pentagon commutes by the pentagon axiom (4.1); the top-left and bottom triangles commute by the triangle axiom (4.2). The left and right

quadrilaterals commute by naturality of a. Since all morphisms invloved are isomorphisns, a diagram chase implies that the top-right triangle commutes. Now, setting $Z = \mathbf{1}$ and using faithfulness of $- \otimes \mathbf{1}$, we conclude that the desired triangle (4.4) commutes. □

Remark 4.6. For any object $X \in \mathcal{C}$, we have the identity

$$l_{\mathbf{1}\otimes X} = \mathbf{1} \otimes l_X. \tag{4.5}$$

Indeed, naturality of $l : \mathbf{1} \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}}$ applied to $l_X : 1 \otimes X \to X$ implies that the square

$$\begin{array}{ccc} \mathbf{1} \otimes (\mathbf{1} \otimes X) & \xrightarrow{l_{\mathbf{1}\otimes X}} & \mathbf{1} \otimes X \\ {\scriptstyle \mathbf{1}\otimes l_X}\downarrow & & \downarrow{\scriptstyle l_X} \\ \mathbf{1} \otimes X & \xrightarrow[l_X]{} & X \end{array}$$

commutes. Since l_X is an isomorphism (in particular a monomorphism), (4.5) holds.

Similarly, it holds that

$$r_{X\otimes\mathbf{1}} = r_X \otimes \mathbf{1}. \tag{4.6}$$

Naturality of $r : - \otimes \mathbf{1} \Rightarrow \mathrm{id}_{\mathcal{C}}$ applied to $r_X : X \otimes \mathbf{1} \to X$ yields the commuting square

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes \mathbf{1} & \xrightarrow{r_{X\otimes\mathbf{1}}} & \mathbf{1} \otimes X \\ {\scriptstyle r_X\otimes\mathbf{1}}\downarrow & & \downarrow{\scriptstyle r_X} \\ \mathbf{1} \otimes X & \xrightarrow[r_X]{} & X \end{array} .$$

The identity (4.6) then follows since r_X is an isomorphism.

We can now prove an identity we may have suspected from Definition 4.1.

Corollary 4.7. *For a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, the identity*

$$r_{\mathbf{1}} = l_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\cong} \mathbf{1}$$

holds.

Proof. Using the triangle axiom (4.2), Remark 4.6 as well as the identity (4.5), we obtain

$$r_{\mathbf{1}} \otimes \mathbf{1} = \mathbf{1} \otimes l_{\mathbf{1}} \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} = l_{\mathbf{1}\otimes\mathbf{1}} \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} = l_{\mathbf{1}} \otimes \mathbf{1}.$$

Now faithfulness of $- \otimes \mathbf{1}$ allows us to conclude. □

Here we provide a different proof of Proposition 4.9 than [2]. In particular, we use Proposition 4.9 as an excuse to introduce a beautiful algebraic tool: the Eckmann-Hilton argument.

Theorem 4.8 (Eckmann-Hilton argument)**.** *Let X be a set equipped with two binary operations $\star, \odot : X \times X \to X$ with two-sided units $e_\star, e_\odot \in X$ respectively. Furthermore, assume that they satisfy the exchange law: for any elements $a, b, c, d \in X$,*

$$(a \star b) \odot (c \star d) = (a \odot c) \star (b \odot d).$$

Then:

(i) the units coincide: $e_\star = e_\odot$

(ii) the laws coincide: $\star = \odot$

(iii) the laws are commutative

(iv) the laws are associative

In particular, X admits a commutative monoid structure.

Proof. The following computation shows that the units coincide:

$$e_\star = e_\star \star e_\star = (e_\odot \odot e_\star) \star (e_\star \odot e_\odot) = (e_\odot \star e_\star) \odot (e_\star \star e_\odot) = e_\odot \odot e_\odot = e_\odot.$$

We then deduce that the laws coincide and are commutative:

$$\begin{aligned} a \star b &= (a \odot e_\odot) \star (e_\odot \odot b) = (a \star e_\odot) \odot (e_\odot \star b) = (a \star e_\star) \odot (e_\star \star b) = a \odot b \\ &= (e_\star \star a) \odot (b \star e_\star) = (e_\star \odot b) \star (a \odot e_\star) = (e_\odot \odot b) \star (a \odot e_\odot) = b \star a. \end{aligned}$$

Finally, we may use the fact that the laws are the same to deduce associativity:

$$a \star (b \star c) = (a \star e_\star) \odot (b \star c) = (a \odot b) \star (e_\star \odot c) = (a \star b) \star (e_\star \star c) = (a \star b) \star c.$$

□

A nice application of the Eckmann-Hilton argument yields the somewhat surprising result below.

Proposition 4.9. *For any locally small monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, $\mathrm{End}_\mathcal{C}(\mathbf{1})$ is a commutative monoid.*

Proof. We denote by $u := l_\mathbf{1} = r_\mathbf{1}$. In addition to the composition of morphisms law on $\mathrm{End}_\mathcal{C}(\mathbf{1})$, we define the an operation $\square : \mathrm{End}_\mathcal{C}(\mathbf{1}) \times \mathrm{End}_\mathcal{C}(\mathbf{1}) \to \mathrm{End}_\mathcal{C}(\mathbf{1})$. Given two endomorphisms $f, g : \mathbf{1} \to \mathbf{1}$, define $g \square f$ to be the unique endomorphism making the square

$$\begin{array}{ccc} \mathbf{1} \otimes \mathbf{1} & \xrightarrow{g \otimes f} & \mathbf{1} \otimes \mathbf{1} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathbf{1} & \xrightarrow[g \square f]{} & \mathbf{1} \end{array}$$

commute. Note that it is well-define because u is an isomorphism. The identity $\mathbf{1} := \mathrm{id}_1 : \mathbf{1} \to \mathbf{1}$ is unital for this operation because for any endomorphism f, naturality of l and r imply that

$$\begin{array}{ccc} \mathbf{1} \otimes \mathbf{1} & \xrightarrow{\mathbf{1} \otimes f} & \mathbf{1} \otimes \mathbf{1} \\ {\scriptstyle l_\mathbf{1}}\downarrow & & \downarrow{\scriptstyle l_\mathbf{1}} \\ \mathbf{1} & \xrightarrow[f]{} & \mathbf{1} \end{array}$$

and

$$\begin{array}{ccc} \mathbf{1} \otimes \mathbf{1} & \xrightarrow{f \otimes \mathbf{1}} & \mathbf{1} \otimes \mathbf{1} \\ {\scriptstyle r_\mathbf{1}}\downarrow & & \downarrow{\scriptstyle r_\mathbf{1}} \\ \mathbf{1} & \xrightarrow[f]{} & \mathbf{1} \end{array}$$

commute; in particular,

$$\mathbf{1} \otimes f = f = f \otimes \mathbf{1}.$$

We now prove that the exchange law relative to the binary operations $\circ$ and $\square$ holds. Given endomorphisms $f, f', g, g' \in \mathrm{End}_\mathcal{C}(\mathbf{1})$, we aim to prove that

$$(g' \circ g) \square (f' \circ f) = (g' \square f') \circ (g \square f).$$

By definition, pasting the two commutative squares yields a commutative rectangle

$$\begin{array}{ccccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{g\otimes f} & \mathbf{1}\otimes\mathbf{1} & \xrightarrow{g'\otimes f'} & \mathbf{1}\otimes\mathbf{1} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \mathbf{1} & \xrightarrow[g\square f]{} & \mathbf{1} & \xrightarrow[g'\square f']{} & \mathbf{1} \end{array} .$$

By functoriality of $\otimes$, we have $(g' \otimes f') \circ (g \otimes f) = (g' \circ g) \otimes (f' \circ f)$, so we have a commutative square

$$\begin{array}{ccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{(g'\circ g)\otimes(f'\circ f)} & \mathbf{1}\otimes\mathbf{1} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \mathbf{1} & \xrightarrow[(g'\square f')\circ(g\square f)]{} & \mathbf{1} \end{array} .$$

By definition, $(g' \circ g)\square(f' \circ f)$ is the unique bottom endomorphism which completes this square; this proves the exchange law.

Now, by Theorem 4.8, the two laws $\circ = \square$ coincide, and are commutative. □

The following proposition, presented in [3], will enable us to construct a fundamental example of a monoidal category.

Proposition 4.10. *For any rings R, S, T, there is a tensor product bifunctor*

$$- \otimes_S - : {}_R\mathbf{Mod}_S \times {}_S\mathbf{Mod}_T \to {}_R\mathbf{Mod}_T.$$

Proof sketch. Given an (R, S)-bimodule M and (S, T)-bimodule N, we define an abelian group

$$M \otimes_S N := \mathbb{Z}[UM \times UN]/K,$$

where U is the forgetful functor to **Set** and K is the subgroup generated by the expressions

- $(x + x', y) - (x, y) - (x', y)$,
- $(x, y + y') - (x, y) - (x, y')$, and
- $(x \cdot s, y) - (x, s \cdot y)$

for all $x, x' \in M$, $y, y' \in N$ and $s \in S$. We denote by $x \otimes y$ the equivalence class of the pair (x, y) for $x \in M$ and $y \in N$, called elementary tensor. Note that by definition of the free abelian group, any element of $M \otimes_S N$ can be written as a finite sum of elementary tensors, which is why it will suffices to define maps from $M \otimes_S N$ only on elementary tensors.

We can upgrade the abelian group $M \otimes_S N$ to an (R, T)-bimodule using the left R-module structure of M and the right T-module structure of N by defining

$$r \cdot (x \otimes y) := (r \cdot x) \otimes y$$

and

$$(x \otimes y) \cdot t := x \otimes (y \cdot t)$$

for all $x \in M$, $y \in N$, $r \in R$ and $t \in T$. This defines the action on objects of $- \otimes_S -$.

For an (R, S)-bimodule homomorphism $f : M \to M'$ and an (S, T)-bimodule homomorphism $g : N \to N'$, there is an (R, T)-bimodule homomorphism $f \otimes_S g : M \otimes_S N \to M' \otimes_S N'$ defined on elementary tensors by

$$(f \otimes_S g)(x \otimes y) := f(x) \otimes g(y)$$

for all $x \in M$ and $y \in N$. This defines the action on morphisms of $- \otimes_S -$. □

Remark 4.11. One can prove that if V, W are k-vector spaces with basis $\mathcal{B}_V$ and $\mathcal{B}_W$ respectively, then

$$\mathcal{B}_V \otimes \mathcal{B}_W := \{v \otimes w \mid v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$$

defines a basis for $V \otimes_k W$.

In particular, if V and W are finite-dimensional, then so is $V \otimes_k W$, and

$$\dim_k V \otimes_k W = \dim_k V \ \times \dim_k W.$$

Monoidal categories arise in various contexts.

Example 4.12. (i) Suppose we have a category $\mathcal{C}$, a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbf{1} \in \mathcal{C}$ so that

$$- \otimes - \circ (- \otimes - \times \mathcal{C}) = - \otimes - \circ (\mathcal{C} \times - \otimes -)$$

and

$$\mathbf{1} \otimes - = \mathrm{id}_{\mathcal{C}} = - \otimes \mathbf{1}.$$

Then choosing the associator, left and right unitors to be identities upgrades the data to a monoidal category, referred to as a *strict monoidal category*.

(ii) For any category $\mathcal{C}$, the functor category $\mathcal{C}^{\mathcal{C}}$ admits a monoidal structure, where the tensor product bifunctor is $\circ : \mathcal{C}^{\mathcal{C}} \times \mathcal{C}^{\mathcal{C}} \to \mathcal{C}^{\mathcal{C}}$. It carries a pair of endofunctors (G, F) to their composite $G \circ F$, and a pair of natural transformations (β, α) to their horizontal composite $\beta * \alpha$ (see Definition 2.34) The exchange law for natural transformations(Lemma 2.36), together with the fact that

$$\mathrm{id}_G * \mathrm{id}_F = \mathrm{id}_{G \circ F}$$

for any $F, G \in \mathcal{C}^{\mathcal{C}}$, prove functoriality of $\circ$. The unit is $\mathrm{id}_{\mathcal{C}} \in \mathcal{C}^{\mathcal{C}}$. Finally, we notice that we are in the setting of Example 4.12 (i), which allows us to conclude.

(iii) Let $\mathcal{C}$ be a category which admits binary products as well as a terminal object $\mathbf{1} \in \mathcal{C}$. By Example 2.74, there is a product bifunctor $- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Moreover, for any triple of objects X, Y, Z, the objects $(X \times Y) \times Z$ and $X \times (Y \times Z)$, together with their projections to X, Y and Z, define limit cones over the $\mathbf{3} = \bullet \ \bullet \ \bullet$-indexed diagram mapping the bullets to the three objects. By Proposition 2.65 and Proposition 2.75, we obtain a unique natural isomorphism

$$a_{X,Y,Z} : (X \times Y) \times Z \xrightarrow{\cong} X \times (Y \times Z)$$

commuting with the legs of the projections maps to the three objects respectively. Moreover, we can see that X equipped with the obvious projections defines a product $l_X : \mathbf{1} \times X \cong X$ and $r_X : X \times \mathbf{1} \cong X$.

To see why the pentagon axiom (4.1) holds, we use the language of representations from section 2.2. For any objects $W, X, Y, Z \in \mathcal{C}$, there is an induced functor $F : \mathbf{4} \to \mathcal{C}$ indexed by the discrete category on 4 objects. Note that each vertex of the pentagon (together with its projections to the four objects) defines a limit of F, and hence a terminal object in the category of cones $\int \mathcal{C}^{\mathbf{4}}(\Delta -, F)$. Moreover, each directed edge of the pentagon defines a morphism between the vertices in the category of cones. Since any two parallel morphisms whose common codomain is a terminal object are equal, we deduce that the pentagon commutes.

The triangle axiom (4.2) can be proven in the same way, applying the theory of representations to a $\mathbf{2}$-indexed diagram.

Dually, via the opposite monoidal category construction from Remark 4.4, any category with binary coproducts and an initial object admits a monoidal structure.

We thus immediately obtain many examples of monoidal categories, including **Set**, **Top**, **Group** and $\mathbf{Vect}_k$ which all admit (binary) products and a terminal object, as well as (binary) coproducts and an initial object. Another example is the category of small categories **Cat**, which admits products (the product of categories) and a terminal object - namely, the terminal category **1**.

(iv) Given a ring R, there is a monoidal category $({}_R\mathbf{Mod}_R, \otimes_R, R, \alpha, \lambda, \rho)$, where $- \otimes_R - : {}_R\mathbf{Mod}_R \times {}_R\mathbf{Mod}_R \to {}_R\mathbf{Mod}_R$ is the tensor product bifunctor introduced in Proposition 4.10. The associator is given by the natural isomorphism

$$\begin{aligned} \alpha_{M,N,P} : (M \otimes_R N) \otimes_R P &\xrightarrow{\cong} M \otimes_R (N \otimes_R P). \\ (x \otimes y) \otimes z &\longmapsto x \otimes (y \otimes z) \end{aligned}$$

The left and right unitors are defined by

$$\begin{aligned} \lambda_M : R \otimes_R M &\xrightarrow{\cong} M \\ r \otimes x &\longmapsto r \cdot x \end{aligned}$$

and

$$\begin{aligned} \rho_M : M \otimes_R R &\xrightarrow{\cong} M \quad . \\ r \otimes x &\longmapsto x \cdot r \end{aligned}$$

It suffices to prove the pentagon axiom (4.1) on elementary tensors, which is a straightforward computation which involves moving around brackets, like for the definition of the associator. The triangle axiom (4.2) is a consequence of the identity

$$x \otimes (r \cdot y) = (x \cdot r) \otimes y$$

for any $x \in M$, $y \in N$ and $r \in R$, which follows from the very definition of the tensor product $M \otimes_R N$.

(v) If R is a commutative ring, any left R-module M automatically inherits a right R-module structure (and thus an (R, R)-bimodule structure) defined by

$$x \cdot r := r \cdot x$$

for any $x \in M$, $r \in R$. Moreover, any left R-module homomorphism $f : M \to N$ is upgraded to an (R, R)-bimodule homomorphism. This observation implies that in this setting, the category ${}_R\mathbf{Mod}$ is isomorphic to the full subcategory of (R, R)-bimodules which arise from a left R-module in the way we have just described. By Remark 4.3, ${}_R\mathbf{Mod}$ thus inherits a monoidal category structure.

In particular, $\mathbf{Vect}_k$ and **Ab** are monoidal categories with their respective tensor products $- \otimes_k - : \mathbf{Vect}_k \times \mathbf{Vect}_k \to \mathbf{Vect}_k$ and $- \otimes_{\mathbb{Z}} - : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$. Moreover, Remark 4.3 and Remark 4.11 imply that $\mathbf{Vect}_k^{\mathrm{fd}}$ is also monoidal.

(vi) For a poset $(P, \leq)$, the data of a monoidal structure is a map $- \otimes - : P \times P \to P$ and an element $\mathbf{1} \in P$ so that

- $x \leq x',\ y \leq y' \implies x \otimes y \leq x' \otimes y'$,
- $(x \otimes y) \otimes z = x \otimes (y \otimes z)$, and
- $\mathbf{1} \otimes x = x = x \otimes \mathbf{1}$.

For instance, the poset $(\mathbb{N}, \leq)$ together with $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $0 \in \mathbb{N}$ defines a monoidal category.

The final definition of this section will be useful to characterize the central concept of section 5.

Definition 4.13. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}, a^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbf{1}_{\mathcal{D}}, a^{\mathcal{D}}, l^{\mathcal{D}}, r^{\mathcal{D}})$ be monoidal categories. A *strong monoidal functor* between them is a triple (F, η, μ), where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor;
- $\eta : F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\cong} \mathbf{1}_{\mathcal{D}}$ is an isomorphism in $\mathcal{D}$;
- μ is a natural isomorphism with domain and codomain functors as displayed:

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times F} & \mathcal{D} \times \mathcal{D} \\ {\scriptstyle -\otimes_{\mathcal{C}} -}\downarrow & \overset{\mu}{\Rightarrow}\ \cong & \downarrow{\scriptstyle -\otimes_{\mathcal{D}} -} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array} .$$

Componentwise:

$$\mu_{X,Y} : F(X \otimes_{\mathcal{C}} Y) \xrightarrow{\cong} F(X) \otimes_{\mathcal{D}} F(X).$$

Moreover, the following associativity and unit coherence diagrams must commute:

$$\begin{array}{ccc} F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F(a^{\mathcal{C}}_{X,Y,Z})} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \\ {\scriptstyle \mu_{X \otimes_{\mathcal{C}} Y, Z}}\downarrow & & \downarrow{\scriptstyle \mu_{X, Y \otimes_{\mathcal{C}} Z}} \\ F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\ {\scriptstyle \mu_{X,Y} \otimes_{\mathcal{D}} F(Z)}\downarrow & & \downarrow{\scriptstyle F(X) \otimes_{\mathcal{D}} \mu_{Y,Z}} \\ (F(X) \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} F(Z) & \xrightarrow[a^{\mathcal{D}}_{F(X),F(Y),F(Z)}]{} & F(X) \otimes_{\mathcal{D}} (FY \otimes_{\mathcal{D}} FZ) \end{array} ,$$

$$\begin{array}{ccc} F(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X) & \xrightarrow{F(l^{\mathcal{C}}_X)} & F(X) \\ {\scriptstyle \mu_{\mathbf{1}_{\mathcal{C}}, X}}\downarrow & & \uparrow{\scriptstyle l^{\mathcal{D}}_{FX}} \\ F(\mathbf{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) & \xrightarrow[\eta \otimes_{\mathcal{D}} F(X)]{} & \mathbf{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) \end{array} ,$$

and

$$\begin{array}{ccc} F(X \otimes_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}) & \xrightarrow{F(r^{\mathcal{C}}_X)} & F(X) \\ {\scriptstyle \mu_{X, \mathbf{1}_{\mathcal{C}}}}\downarrow & & \uparrow{\scriptstyle r^{\mathcal{D}}_{FX}} \\ F(X) \otimes_{\mathcal{D}} F(\mathbf{1}_{\mathcal{C}}) & \xrightarrow[F(X) \otimes_{\mathcal{D}} \eta]{} & F(X) \otimes_{\mathcal{D}} \mathbf{1}_{\mathcal{D}} \end{array} .$$

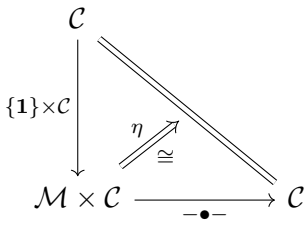
5 Actegories

In section 4, we have categorified the notion of a monoid. We now categorify the notion of an *action* of a monoid on a set.

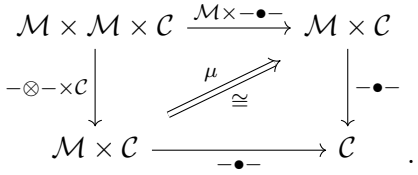
Definition 5.1. Let $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. A *left $\mathcal{M}$-category* is a category $\mathcal{C}$ equipped with an *action bifunctor*

$$- \bullet - : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$

and two natural isomorphisms (called *unitor* and *multiplicator* respectively)



and



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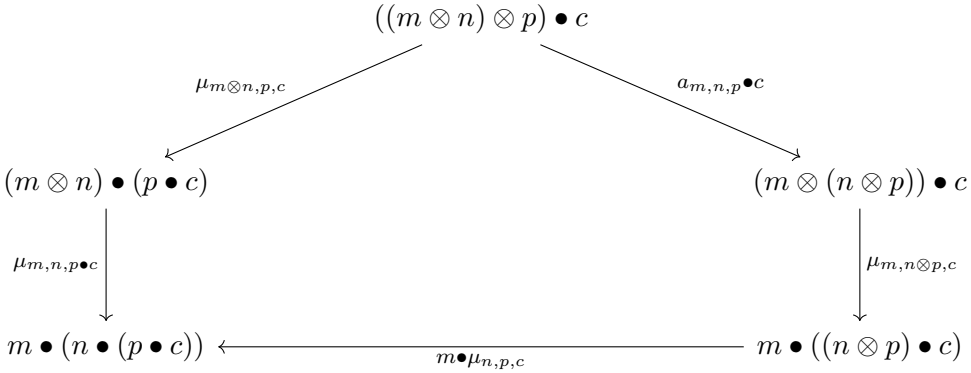
Compentwise:

$$\eta_c : \mathbf{1} \bullet c \xrightarrow{\cong} c$$

and

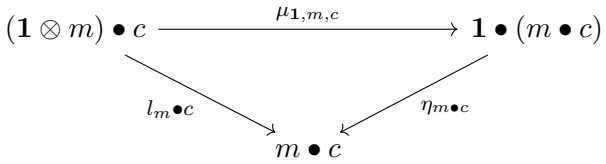
$$\mu_{m,n,c} : (m \otimes n) \bullet c \xrightarrow{\cong} m \bullet (n \bullet c)$$

for all $m, n \in \mathcal{M}$ and $c \in \mathcal{C}$. Moreover, the pentagon



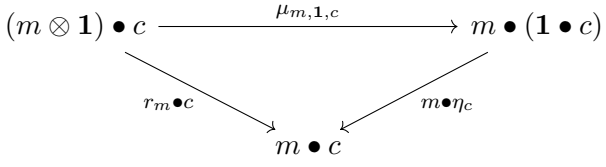
(5.1)

as well as the two triangles



(5.2)

and



(5.3)

are required to commute for all $m, n, p \in \mathcal{M}$ and $c \in \mathcal{C}$.

Unpacking Definition 4.13 and Definition 5.1, we see that the isomorphism (2.1) obtained by currying yields a characterization of the notion of left actions of monoidal categories.

There are many examples of actegories; the ones we expose generally use properties of categories, functors and natural transformations.

Proposition 5.2. *For any monoidal category $\mathcal{M}$ and ordinary category $\mathcal{C}$, there is a bijection*

$$\{\textit{left } \mathcal{M}\textit{-actions } \mathcal{M} \times \mathcal{C} \xrightarrow{-\bullet-} \mathcal{C}\} \leftrightarrow \{\textit{strict monoidal functors } \mathcal{M} \to \mathcal{C}^{\mathcal{C}}\}$$

with respect to the monoidal stucture of $\mathcal{C}^{\mathcal{C}}$ seen in Example 4.12 (ii).

Example 5.3. (i) For any monoidal category $\mathcal{M}$ and category $\mathcal{C}$, there is a trivial action $\pi_{\mathcal{C}} : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$, with the unitor η and multiplicator μ being identities.

(ii) Any monoidal category $\mathcal{M}$ acts on itself; the action bifunctor is the tensor product bifunctor $- \otimes - : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, the unitor η is the left unitor l and the multiplicator μ is the associator a. The diagrams (5.1), (5.2) and (5.3) in Definition 5.1 that are required to commute correspond respectively to the pentagon axiom (4.1), the triangle (4.3) and the triangle (4.2).

(iii) For any category $\mathcal{C}$, there is an evaluation action

$$ev : \mathcal{C}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{C}$$

which carries a pair (F, c) to Fc, and a pair of morphisms $(F, c) \xrightarrow{(\alpha, f)} (G, d)$ to the common composite (by naturlaity of α)

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & \overset{ev_{(\alpha,f)}}{\searrow} & \downarrow{\scriptstyle Gf} \\ Fd & \xrightarrow[\alpha_d]{} & Gd \end{array} .$$

It is easy to see that $ev(\mathrm{id}_{\mathcal{C}}, -) : \mathcal{C} \to \mathcal{C}$ is the identity; in particular, we may chose η to be the identity natural transformation. We would like to make the same choice for μ, so we need to first verify that

$$\begin{array}{ccc} \mathcal{C}^{\mathcal{C}} \times \mathcal{C}^{\mathcal{C}} \times \mathcal{C} & \xrightarrow{\mathcal{C}^{\mathcal{C}} \times ev} & \mathcal{C}^{\mathcal{C}} \times \mathcal{C} \\ {\scriptstyle -\circ- \times \mathcal{C}}\downarrow & & \downarrow{\scriptstyle ev} \\ \mathcal{C}^{\mathcal{C}} \times \mathcal{C} & \xrightarrow[ev]{} & \mathcal{C} \end{array} \tag{5.4}$$

commutes. Note that a morphism $(G, F, c) \xrightarrow{(\beta, \alpha, f)} (G', F', c') \in \mathcal{C}^{\mathcal{C}} \times \mathcal{C}^{\mathcal{C}} \times \mathcal{C}$ induces a commutative cube

$$\begin{array}{ccccc} GFc & & \xrightarrow{G\alpha_c} & & GF'c \\ & \searrow^{\beta_{Fc}} & & & \downarrow{\scriptstyle GF'f} \searrow^{\beta_{F'c}} \\ {\scriptstyle GFf}\downarrow & & G'Fc & \xrightarrow{G'\alpha_c} & G'F'c \\ & & {\scriptstyle G'Ff}\downarrow & & \downarrow \\ GFc' & & \xrightarrow{G\alpha_{c'}} & GF'c' & \downarrow{\scriptstyle G'F'f} \\ & \searrow^{\beta_{Fc'}} & & \searrow^{\beta_{F'c'}} & \\ & & G'Fc' & \xrightarrow{G'\alpha_{c'}} & G'F'c' \end{array} . \tag{5.5}$$

When evaluating this morphism along the bottom-left composite in the square (5.4), the morphism $GFc \to G'F'c'$ thus obtained is either of the four composites in the cube (5.5): the top square followed by $G'F'f$, or β_{Fc} followed by the front square. Doing the same for the top-right composite in square (5.4), the resulting morphism $Gfc \to G'F'c'$ in the cube is again β_{Fc} followed by the front square, or the back square followed by $\beta_{F'c'}$. The commutative cube (5.5) implies that these composites are all equal, proving that the square (5.4) commutes. We can now choose μ to be the identity natural transformation.

(iv) For any categories $\mathcal{C}$ and $\mathcal{D}$, there is a precomposition action

$$pre : (\mathcal{C}^{\mathcal{C}})^{rev} \times \mathcal{D}^{\mathcal{C}} \to \mathcal{D}^{\mathcal{C}};$$

a pair (T, F) is mapped to FT, and a morphism $(\beta, \alpha) : (F, T) \to (F', T')$ to the horizontal composite $\alpha * \beta : FT \to F'T'$ (notice that order is reversed!). Functoriality follows from the exchange law for natural transformations (Lemma 2.36). Moreover, we may choose η and μ to be identities; checking that the suitable functor composites are equal follows from usual calculus of functor compositon and horizontal composition of natural transformations.

(v) There is an action

$$(-)^{-} : \mathbf{Cat}^{op} \times \mathbf{CAT} \to \mathbf{CAT}$$

that carries a pair $(\mathcal{I}, \mathcal{C})$ to the functor category $\mathcal{C}^{\mathcal{I}}$, and a pair of morphisms $(F : \mathcal{J} \to \mathcal{I}, G : \mathcal{C} \to \mathcal{D})$ to either composite functor

$$\begin{array}{ccc} \mathcal{C}^{\mathcal{I}} & \xrightarrow{G_*} & \mathcal{D}^{\mathcal{I}} \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ \mathcal{C}^{\mathcal{J}} & \xrightarrow[G_*]{} & \mathcal{D}^{\mathcal{J}} \end{array} .$$

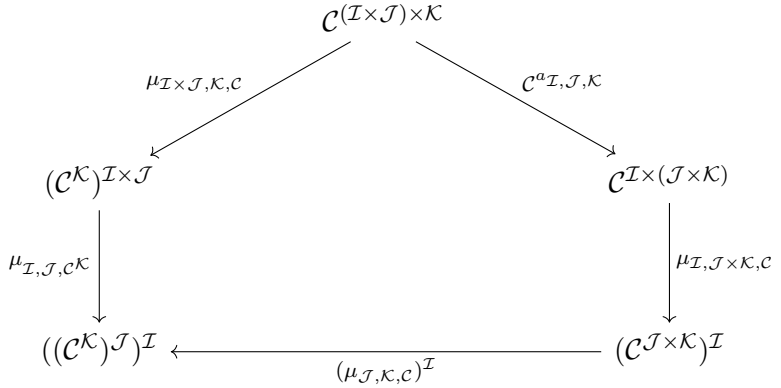
This time we cannot choose the unitor and multiplicator to be identities. Instead, for every $\mathcal{C} \in \mathbf{CAT}$, the unitor is defined by

$$\eta_{\mathcal{C}} : \mathcal{C}^{\mathbf{1}} \xrightarrow{\cong} \mathcal{C},$$

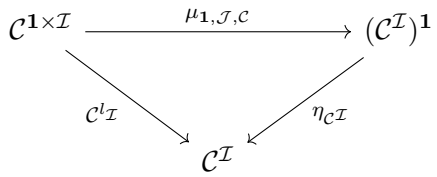
the isomorphism of categories which views an object $c \in \mathcal{C}$ as a functor $c : \mathbf{1} \to \mathcal{C}$, and a morphism $f : c \to c'$ as a natural transformation between the corresponding functors $f : c \Rightarrow c'$. The components of the multiplier

$$\mu_{\mathcal{I},\mathcal{J},\mathcal{C}} : \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \xrightarrow{\cong} (\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}$$

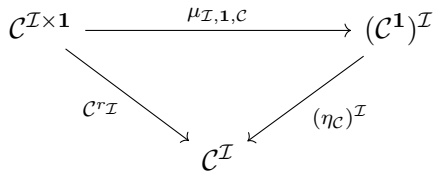
are defined by currying (see Example 2.38) for any $\mathcal{I}, \mathcal{J} \in \mathbf{Cat}$ and $\mathcal{C} \in \mathbf{CAT}$. The coherence axioms are verified because the pentagon



as well as the two triangles



and



commute for any $\mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathbf{Cat}$ and $\mathcal{C} \in \mathbf{CAT}$.

(vi) For a topological space $(X, \mathcal{T}) \in \mathbf{Top}$, there is a monoidal poset category $(\mathcal{T}, \cap, X)$ by Example 4.12 (vi), and an action on the slice category (defined in Example 2.58 (iii)) $\mathbf{Top}/(X, \mathcal{T})$ over $(X, \mathcal{T})$

$$\mathcal{T} \times \mathbf{Top}/(X, \mathcal{T}) \to \mathbf{Top}/(X, \mathcal{T}).$$

It maps a pair $(U, f : (Y, \mathcal{T}') \to (X, \mathcal{T}))$ to $f|_{f^{-1}(U)} : (f^{-1}(U), \mathcal{T}_{f^{-1}(U)}) \to (X, \mathcal{T})$, and a morphism $(U, f : (Y, \mathcal{T}') \to (X, \mathcal{T})) \xrightarrow{(i,h)} (V, g : (Z, \mathcal{T}'') \to (X, \mathcal{T}))$ - that is, a continuous map $h : (Y, \mathcal{T}) \to (Z, \mathcal{T}'')$ so that $f = g \circ h$, in addition to the condition that $U \subset V$ - to the morphism $h|^{g^{-1}(V)}_{f^{-1}(U)} : (f^{-1}(U), \mathcal{T}'_{f^{-1}(U)}) \to (g^{-1}(V), \mathcal{T}''_{g^{-1}(V)}) \in \mathbf{Top}/(X, \mathcal{T})$; it is well-defined (i.e. $f|_{f^{-1}(U)} = g|_{g^{-1}(V)} \circ h|^{g^{-1}(V)}_{f^{-1}(U)}$) precisely because $U \subset V$. Finally, we choose η and μ to be identities.

6 Conclusion

In this report, we exposed fundamental concepts from category theory; more precisely, those that are required to fully grasp the concept of a monad. We then explored the connection between monads and adjunctions, emphasizing the ability of the former to encode algebraic structure in a purely categorical way. We then lifted the concept of a monoid and monoid actions to the more abstract, categorical framework of monoidal categories and their actions on arbitrary categories.

This work was carried out as part of the Summer in the Lab (SIL) program, whose support I gratefully acknowledge. I would like to thank Professor Kathryn Hess Bellwald and the Laboratory for Topology and Neuroscience at EPFL for making this project possible. I am especially grateful to Professor Hess for her guidance and dedication, for meeting with me regularly, and for closely following the development of my work throughout the project.

Regarding the use of artificial intelligence, it was employed solely to assist with the formatting of larger commutative diagrams by providing LaTeX templates.

On a personal level, I found it particularly exciting to discover this fascinating subject, having had almost no prior exposure to category theory before the internship. I especially enjoyed the various mathematical excursions that arose along the way and the opportunity to explore certain topics in greater depth. These experiences gave me a first taste of the research process: formulating questions, investigating them, and trying to develop satisfactory answers.

The initial objective of this internship was to understand the machine that produces monads from actegories, explained in the seminar talk [5]. Unfortunately, learning about category theory turned out to take up most of my time!

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